INITIAL ENLARGEMENT OF FILTRATIONS AND THE DUFRESNE IDENTITY

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These notes are based on lecture notes by Jeanblanc, Nikeghbali, and Mansuy and Yor along with papers by Matsumoto and Yor and O’Connell and Yor. Citations are included at the end.

The usual interpretation of a filtration $\mathcal{F}_t$ of $\sigma-$algebras is that they represent the developing knowledge of an actor as time changes. In applications and theory, the typical objects of study are (semi)martingales, so it is a natural question to ask what happens to a (semi)martingale if we alter the $\sigma-$algebra we are working with. To motivate this study, we consider a classical example:

**Brownian Bridge.**

Typically, we define the Brownian Bridge to be the process $b_t = (B_t \mid 0 \leq t \leq 1 \mid B_1 = 0)$. This definition can be made rigorous by observing that

$$B_t = (B_t - tB_1) + tB_1$$

and that $(B_t - tB_1)$ and $B_1$ are independent Gaussian random variables (by computing covariances). If we condition on $|B_1| < \epsilon$ and take $\epsilon \to 0$ we can define $b_t$ to be the weak limit as $\epsilon \to 0$ of

$$((B_t - tB_1) + tB_1 \mid 0 \leq t \leq 1 \mid |B_1| < \epsilon)$$

and conclude that in law

$$(b_t \mid 0 \leq t \leq 1) = (B_t - tB_1 \mid 0 \leq t \leq 1)$$

As an alternative, we can forgo the weak limit and directly consider what happens to the Brownian motion $B_t$ if we extend its filtration to

$$\mathcal{F}_t^{(B_1)} = \mathcal{F}_t \vee \sigma(B_1) = \cap_{\epsilon > 0}(\mathcal{F}_{t+\epsilon} \vee \sigma(B_1))$$

where the second equality above is there to highlight that $\mathcal{F}_t^{(B_1)}$ still satisfies the usual conditions. Notice that $B_t$ is no longer a $\mathcal{F}_t^{(B_1)}$ martingale since

$$E[B_1|\mathcal{F}_t^{(B_1)}] = B_1 \neq B_t$$

We claim that

$$\beta_t = B_t - \int_0^t \frac{B_1 - B_s}{1 - s} ds$$

$(0 \leq t \leq 1)$ is a martingale with respect to the filtration $\mathcal{F}_t^{(B_1)}$. Notice that since $B_1 - B_s$ has the same distribution as $N(0, 1) \cdot \sqrt{1 - s}$ it makes sense to condition
\( \beta_t \) on \( \mathcal{F}_s^{(B_1)} \). Observe that for \( s \leq t \leq 1 \) we have \( \mathcal{F}_t^{(B_1)} = \mathcal{F}_t \cup \sigma(B_1 - B_s) \). Independence of \( \mathcal{F}_s \) from \( \sigma(B_{s+h} - B_s, h \geq 0) \) gives that we have

\[
E[B_t - B_s | \mathcal{F}_s^{(B_1)}] = E[B_t - B_s | \mathcal{F}_s \cup \sigma(B_1 - B_s)]
\]

\[
= E[B_t - B_s | \mathcal{F}_t] = \frac{t-s}{1-s} (B_1 - B_s)
\]

To see this, observe that

\[
\begin{pmatrix}
X \\
Y
\end{pmatrix} = \begin{pmatrix}
B_t - B_s \\
B_1 - B_s
\end{pmatrix}
\]

is a joint Gaussian and can be written in distribution

\[
\begin{pmatrix}
X \\
Y
\end{pmatrix} = \begin{pmatrix}
\sqrt{t-s} & 0 \\
\sqrt{t-s} & \sqrt{1-t}
\end{pmatrix} \begin{pmatrix}
Z_1 \\
Z_2
\end{pmatrix}
\]

where \( Z_1 \) and \( Z_2 \) are independent standard normal random variables. This is a normal distribution with mean zero and covariance matrix

\[
\Sigma = \begin{pmatrix}
t-s & t-s \\
t-s & 1-s
\end{pmatrix}
\]

Under these definitions, we would like to compute \( E[X|Y] \). This is a 431 level computation which gives our desired conclusion. We have

\[
\beta_t - \beta_s = B_t - B_s + \left[ \int_0^s \frac{B_1 - B_u}{1-u} du - \int_0^t \frac{B_1 - B_u}{1-u} du \right]
\]

\[
= B_t - B_s - \int_s^t \frac{B_1 - B_u}{1-u} du
\]

So since we know that

\[
E[B_t - B_s | \mathcal{F}_s^{(B_1)}] = \frac{t-s}{1-s} (B_1 - B_s)
\]

We then have

\[
E\left[ \int_s^t \frac{B_1 - B_u}{1-u} du | \mathcal{F}_s^{(B_1)} \right] =
\]

\[
= \int_s^t E \left[ \frac{B_1 - B_u}{1-u} | \mathcal{F}_s^{(B_1)} \right] du
\]

\[
= \int_s^t \frac{B_1 - B_s - E[B_u - B_s | \mathcal{F}_s^{(B_1)}]}{1-u} du
\]

\[
= \int_s^t \frac{B_1 - B_u}{1-u} - \frac{u-s}{1-s} (B_1 - B_s) du
\]

\[
= \frac{t-s}{1-s} (B_1 - B_s)
\]
Therefore $\beta_t$ is a martingale. We can compute the quadratic variation of

$$\beta_t = B_t - \int_0^t \frac{B_1 - B_s}{1 - s} ds$$

to be 1 by observing that the only contribution comes from $B_t$ since $\int_0^t \frac{B_1 - B_s}{1 - s} ds$ is finite variation. Therefore $\beta_t$ is Brownian Motion with respect to $\mathcal{F}_t^{(B_1)}$. The semimartingale decomposition of $B_t$ is then

$$B_t = \beta_t + \int_0^t \frac{B_1 - B_s}{1 - s} ds.$$  

A particular consequence of the above is that the standard Brownian bridge is a solution to the stochastic differential equation [Recall that here $B_t$ is a 'generalized Brownian bridge' while $\beta_t$ is a Brownian motion under the new filtration, so on the set where $B_1 = 0$ we have what follows]

$$\begin{cases}
  db_t = -\frac{b_t}{1-t}dt + dW_t & 0 \leq t < 1 \\
  b_0 = 0
\end{cases}$$

which gives the characterization

$$b_t = (1 - t) \int_0^t \frac{1}{1 - s} dW_s.$$
Initial Enlargements of Brownian Filtrations.

Although there are many different ways to enlarge a filtration, most of the research to date has focused on two special cases:

\[ F_t^{(L)} = \cap_{\epsilon > 0} (F_{t+\epsilon} \vee \sigma(L)) \]

for some real valued random variable \( L \) or

\[ G_t = \cap_{\epsilon > 0} (F_{t+\epsilon} \vee \sigma(1_{\tau \leq t+\epsilon})) \]

for some random time \( \tau \). Notice that in general, a \( F_t \) martingale need not be a \( F_t^{(L)} \) semimartingale. For an example, observe that if \( F_t \) is the natural filtration of a Brownian Motion, then \( F_t \) is separable so it follows that \( F_\infty \) is generated by some bounded random variable \( L \).

But then observe that

\[ F_{L+t}^{(L)} = F_\infty \]

and therefore \( F_t^{(L)} \) martingales are constant after the (bounded) stopping time \( L \) and so, for example, \( B_t \) is not a semimartingale with respect to \( F_t^{(L)} \). There are some conditions on the random variable \( L \) we use to ensure that \( F_t \) martingales remain \( F_t^{(L)} \) semimartingales.

All the results here hold in much greater generality, but given the length of the talk, I am just going to work with Brownian filtrations \( F_t \) and random variables satisfying strong conditions. This section follows Mansuy-Yor and Nikéghbali. Call \( \lambda_t(f) = E[f(X)|F_t] \) where we choose \( \lambda_t(f) \) to be the continuous version of this process.

We introduce the process \( \hat{\lambda}_t(f) \) which satisfies

\[ \lambda_t(f) = E[f(X)] + \int_0^t \hat{\lambda}_s(f) dB_s \]

which exists since \( F_t \) is a Brownian filtration.

Mansuy-Yor [2] claims that a technical proof shows that there exists a predictable family of measures \((\lambda_t(dx), t \geq 0)\) so that

\[ \lambda_t(f) = \int f(x) \lambda_t(dx) \]

and we assume that there exists a predictable family of measures \((\hat{\lambda}_t(dx), t \geq 0)\) so that

\[ \hat{\lambda}_t(f) = \int f(x) \hat{\lambda}_t(dx) \]

where \( dPds \) almost surely \( \hat{\lambda}_t \ll \lambda_t \) with \( \hat{\lambda}_t(dx) = \rho(x,t) \lambda_t(dx) \). Then we have the following theorem called “Yor’s Method”:

\[ ^1\text{There is a proof of this in the appendix.} \]
Theorem. For any \( \mathcal{F}_t \)-martingale \( M_t = \int_0^t m_s dB_s \), there exists \( \tilde{M}_t \), a \( \mathcal{F}_t^{(X)} \) local martingale, such that

\[
M_t = \tilde{M}_t + \int_0^t \rho(X,s)d < M, B >_s
\]

\[
= \tilde{M}_t + \int_0^t \rho(X,s)m_s ds
\]

provided that

\[
\int_0^t |p(X,s)||d < M, B >_s| < \infty
\]

almost surely.

Proof. Idea: Let \( f \) be any test function, \( X \) be a random variable satisfying the above assumptions \( M_t \) be a \( \mathcal{F}_t \) martingale and \( \Lambda_s \in \mathcal{F}_s \) with \( s < t \). We would like to show that \( M_t \) is a semimartingale with respect to \( \mathcal{F}_t^{(X)} \) and show that

\[
E[1_{\Lambda_s} f(X)(M_t - M_s)] = E[1_{\Lambda_s} f(X) \int_s^t \rho(u,L)d < X, B >_u]
\]

This follows from the fact that

\[
E[1_{\Lambda_s} f(X)(M_t - M_s)] = E[1_{\Lambda_s} \lambda_t(f)M_t - \lambda_s(f)M_s]
\]

\[
= E[1_{\Lambda_s} \langle \lambda(f), M \rangle_t - \langle \lambda(f), M \rangle_s]
\]

where we have used the identity (since all martingales here are continuous we can drop left limits)

\[
\lambda_t(f)M_t - \lambda_s(f)M_s = \int_s^t \lambda_t(f)dM_t + \int_s^t M_t d\lambda_t(f) + \langle \lambda(f), M \rangle_t - \langle \lambda(f), M \rangle_s
\]

and the fact that all of those stochastic integrals are zero after conditioning on \( \mathcal{F}_s \). Then this is

\[
= E[1_{\Lambda_s} \int_s^t \lambda(f)u d\langle B, M \rangle_u]
\]

\[
= E[1_{\Lambda_s} \int_s^t \int_\mathbb{R} f(x)\rho(x,u)\lambda_u(dx)d\langle B, M \rangle_u]
\]

\[
= E[1_{\Lambda_s} f(X) \left( \int_0^t \rho(X,u)d\langle B, M \rangle_u - \int_0^s \rho(X,u)d\langle B, M \rangle_u \right)]
\]

The last equality is easiest to see if we replace \( d\langle M, B \rangle_u \) with \( m_u ds \) and use that

\[
\int_\mathbb{R} f(x)\rho(x,u)\lambda_u(dx) = E[f(X)\rho(X,u)|\mathcal{F}_u]
\]

This verifies the result on the product of test functions and indicators, so now we need to make a monotone class argument to extend the result.

\[\square\]
**Example.** (Brownian Bridge) Observe that for a test function $g$ by the Markov property we have

$$E[g(B_1)|\mathcal{F}_t] = E[g(B_1 - B_t + B_t)|\mathcal{F}_t] = \int_{\mathbb{R}} g(x)p(1-t, x, B_t)dx$$

where $p(t, x, y) = \frac{1}{\sqrt{2\pi t}} e^{-\frac{(x-y)^2}{2t}}$ is the transition function for Brownian Motion. It follows then that

$$\lambda_t(dx) = \frac{1}{\sqrt{2\pi(1-t)}} e^{-\frac{(x-B_t)^2}{2(1-t)}} dx$$

satisfies

$$E[g(B_1)|\mathcal{F}_t] = \int_{\mathbb{R}} g(x)\lambda_t(dx)$$

To find $\hat{\lambda}_t(dx)$ we can differentiate $\lambda_t(dx)$ with respect to $B_t$ (since this is a martingale, there is no quadratic variation) to find that

$$\hat{\lambda}_t(dx) = \frac{x - B_t}{1-t} \lambda_t(dx)$$

and thus

$$B_t = \tilde{B}_t + \int_0^t \frac{B_1 - B_s}{1-s} ds$$

where $\tilde{B}_t$ is a $\mathcal{F}_t^{(X)}$ martingale (and, in fact, Brownian Motion by Levy’s criterion).

**Example.** This example comes from Jeanblanc [1]. Let $B_t$ be standard Brownian Motion and consider $X = \int_0^\infty f(s)dB_s$ where $\int_0^\infty f^2(s)ds < \infty$. Then we can compute the semimartingale decomposition of $B_t$ in $\mathcal{F}_t^{(X)}$ by observing that conditional on $\mathcal{F}_t$ $X$ is Gaussian with mean $m_t = \int_0^t f(s)dB_s$ and variance $\sigma_t^2 = \int_0^\infty f^2(s)ds$

$$E[\varphi(X)|\mathcal{F}_t] = \int_{\mathbb{R}} \varphi(x) \frac{1}{\sqrt{2\pi t}} e^{-\frac{(x-m_t)^2}{2\sigma_t^2}} dx$$

so our $\lambda_t(dx) = \frac{1}{\sqrt{2\pi t}} e^{-\frac{(x-m_t)^2}{2\sigma_t^2}} dx$.

So long as the integrability condition above is satisfied, we can compute the stochastic differential of the kernel to find $\hat{\lambda}_t(dx)$. Again, we know that this is a martingale, so we can ignore the nonstochastic terms. We find that

$$\hat{\lambda}_t(dx) = f(t)\frac{(x - m_t)}{\sigma_t^2} \frac{1}{\sqrt{2\pi t}} e^{-\frac{(x-m_t)^2}{2\sigma_t^2}} dx$$

so

$$\rho(x, t) = f(t)\frac{(x - m_t)}{\sigma_t^2}$$
Then assuming that $\int_0^\infty \frac{|f(t)|}{\sigma_t^2} dt < \infty$ it follows that the semimartingale decomposition of $B_t$ is

$$B_t = \hat{B}_t + \int_0^t \frac{f(s)}{\sigma_s^2} \int_s^\infty f(u) dB_u ds$$
**Geometric Brownian Motion, stocks, and the Asian Option.** In what follows, we will consider a relationship between Brownian motion of drift $\mu$ and Brownian motion with drift $-\mu$, as well as the Dufresne relation. Both of these results will involve considering the process

$$A_t(\mu) = \int_0^t e^{2B_s(\mu)} \, ds$$

where $B_t(\mu)$ is Brownian motion with drift $\mu$. Before moving on, it seems appropriate to motivate the study of these objects from an applied perspective, since these results (particularly the Dufresne identity) originally came from mathematical finance.

In finance, a naive approach to modelling stocks is to assume that they follow a simple random walk or Brownian Motion (this is the first model introduced in the economics sequence I took). This model has the advantage of being reasonably well understood and largely 'solvable' in that the objects of interest have universal limits in terms of Brownian Motion. The downside of this model is that it is observably false: stock prices do not (and essentially can not) become negative. A natural response to this problem is to consider the 'closest' thing to a positive Brownian motion—the geometric Brownian Motion. The assumption that stocks follow Geometric Brownian Motion is common in the finance literature that I have seen, though that is a fairly small sample. There are some other justifications, but as far as I can tell the assumption is not a particularly accurate one.

If we do assume that stock prices do follow Geometric Brownian Motion with some drift then the $A_t(\mu)$ introduced above is a natural object of study, since $1/2A_t(\mu)$ is the time average of the stock price—which is something actually traded on the exchanges and is called an Asian Option. This relationship was the reason Dufresne originally wanted to study the distribution of $A_t(\mu)$.

As further motivation, there is a form of generalized $MM1$ queue studied by O’Connell and Yor where the queue length is a function of $A_t(\mu)$.
Lamperti’s Relation. A final piece of setup is worth mentioning before getting to the main results. I will not be proving the decomposition formulae in what follows, since the computations I was able to find are involved and require a rather detailed knowledge of Bessel processes that is not the focus of this talk. The connection between \( A_t^{(\mu)} \), Geometric Brownian Motion, and Bessel processes goes through a result known as Lamperti’s relation, which in this case specializes to the equivalence

\[
e^{2B_t^{(-\mu)}} = R_t^{(-\mu)} A_t^{(-\mu)}
\]

where \( R_t^{(\mu)} \) is a Bessel process with drift \( \mu \). With this, we can obtain that \( A^{(-\mu)}_\infty \) has the same distribution as \( T_0 = \inf \{ s : R_s^{(-\mu)} = 0 \} \), so we use known results about the joint distribution of a Bessel process and its zeroes to find results about \( A^{(-\mu)}_\infty \).

We will consider the semi-martingale decomposition of \( B_t^{(-\mu)} \) in the enlarged filtration \( \mathcal{F}_t^{(A^{(-\mu)}_\infty)} \). A computation which can be found in Mansuy-Yor as exercise 17[2] (which I will not do) gives that

\[
\rho(x, t) = 2\mu - \frac{e^{2B_t^{(-\mu)}}}{x - A_t^{(-\mu)}}
\]

which will give us the semimartingale decomposition we will be looking for.

It is also possible to produce this decomposition directly through a combination of Lamperti’s relation (which appears in the proof of the previous statement as well) and some information about conditioned Bessel processes. O’Connel and Yor [5] cite a book by Yor I was unable to find a copy of for a proof that \( R_u^{(-\mu)} \) conditioned on the first hitting time of zero, \( T_0 = t \) solves

\[
r(u) = 1 + \beta_u + (1 + \mu) \int_0^u \frac{ds}{r(s)} - \int_0^u \frac{r(s)ds}{t - s}
\]

where \( \beta_u \) is Brownian Motion; that is

\[
R_u^{(-\mu)} = 1 + \beta_u + (1 + \mu) \int_0^u \frac{ds}{R_s^{(-\mu)}} - \int_0^u \frac{R_s^{(-\mu)}ds}{T_0 - s}
\]

which is the semi-martingale decomposition of \( R_u^{(-\mu)} \) in the filtration enlarged by \( T_0 \). Evaluating at \( u = A_t^{(-\mu)} \) and changing variables gives the previous semi-martingale decomposition. This is result is the conclusion of equations (3.7) and (3.8) in Matsumoto-Yor.
Brownian Motions with Different Drifts.

**Theorem.** (Matsumoto Yor 2001) There is a random variable $\gamma_\mu$ independent of $B^{(\mu)}$ so that for

$$A^{(\mu)}_t = \int_0^t e^{2B^{(\mu)}_s} \, ds$$

and

$$\hat{B}^{(\mu)}_t = B^{(\mu)}_t - \log(1 + 2\gamma_\mu A^{(\mu)}_t)$$

we have

$$\{B^{(-\mu)}_t, \ t \geq 0\} \sim \{\hat{B}^{(\mu)}_t, \ t \geq 0\}$$

**Proof.** Let $\mathcal{F}^{(\mu)}_t = \sigma(B^{(\mu)}_s, \ s \leq t)$ and set $\hat{\mathcal{F}}^{(-\mu)}_t = \mathcal{F}^{(\mu)}_t \vee \sigma(A^{(-\mu)}_\infty)$. The semimartingale decomposition for initial enlargements of filtration gives that there exists a $\hat{\mathcal{F}}^{(\mu)}_t$ – Brownian Motion $\hat{W}_t$ independent of $A^{(-\mu)}_\infty$ (independent because $A^{(-\mu)}_\infty$ is $\hat{\mathcal{F}}^{(\mu)}_0$ – measurable) with

$$B^{(-\mu)}_t = W_t + \mu t - \int_0^t \frac{e^{2B^{(-\mu)}_s}}{(A^{(-\mu)}_\infty - A^{(-\mu)}_s)} \, ds$$

As a comment, proving that this formula is correct is nontrivial, though not incredibly difficult. Exercise 17 in section 1.5 of Mansuy-Yor [2] outlines a proof. Most of the details of that computation can be found in O’Connell and Yor [5].

Under the condition that $A^{(-\mu)}_\infty = \frac{1}{2c}$ (recall that this random variable is measurable with respect to our filtration so this is just working $\omega$ by $\omega$). Then we find that $(B^{(-\mu)}_t \ t \geq 0)$ is a solution of

$$z_t(\omega) = W_t(\omega) + \mu t - \int_0^t \frac{e^{2z_s(\omega)}}{(1/2c - \int_0^s e^{2z_u(\omega)} \, du)} \, ds$$

with initial data $(W_t(\omega) + \mu t, \ t \geq 0)$.

It is an analytic fact that functionals of this form have unique solutions (see Matsumoto and Yor [3] for a proof and, in fact, a formula) and we can check that

$$z_t(\omega) = W^{(\mu)}_t(\omega) - \log(1 + 2cA^{(\mu)}_t(\omega))$$

is such a solution, where

$$W^{(\mu)}_t = W_t + \mu t$$

and

$$A^{(\mu)}_t = \int_0^t e^{2W_s} \, ds$$

$\square$
The Dufresne Identity.

Theorem. (Dufresne, proof based on O’Connell-Yor 2001) Let $\mu > 0$ and let $B_t^{(\mu)}$ be a Brownian Motion with drift $\mu$. Then

$$A_s^{(-\mu)} = \frac{1}{2\gamma_\mu}$$

where $\gamma_\mu$ is a gamma distribution with parameter $\mu$ independent of $B_t^{(\mu)}$.

Proof. Consider the diffusion process

$$X_t^\mu = e^{-2B_t^\mu} A_t^{(\mu)} = \int_0^t e^{2(B_s^{(\mu)} - B_t^{(\mu)})} ds$$

$$= \int_0^t e^{2(B_s - B_t + \mu(s-t))} ds$$

and notice that as $t \to \infty$ this converges to a stationary distribution since a change of variables gives that the above has the same distribution as

$$\int_0^t e^{2Bu - \mu u} du$$

we find that $A_s^{(-\mu)}$ has the desired stationary distribution.

Now $X_t$ has Ito equation given by

$$dX_t = dt - 2X_t dB_t^{(\mu)} + 2X_t d[B_t^{(\mu)}]$$

$$= dt - 2X_t[\mu t + dB_t] + 2X_t dt$$

$$= [1 + 2X_t(1 - \mu)] dt + 2X_t dB_t$$

which has generator

$$\mathcal{L} = 2x^2 \frac{d^2}{dx^2} + [1 + 2x(1 - \mu)] \frac{d}{dx}$$

and adjoint

$$\mathcal{L}^* \rho(x) = \frac{d^2}{dx^2} 2x^2 \rho(x) - \frac{d}{dx} [1 + 2x(1 - \mu)] \rho(x)$$

Since $A_s^{(-\mu)}$ is supported on $(0, \infty)$, we know that $\rho(x) = 0$ for $x \leq 0$ a.e. and for $x > 0$ we may compute that $\rho(x) = x^{1-\mu} e^{-\frac{1}{2\mu}}$ solves the above adjoint equation. There is a slight technical problem here in that there is another solution to the above equation, but the other function has a nonzero limit at zero and so since we know that the distribution of $A_s^{(\mu)}$ must decay at least exponentially at zero it cannot factor into the stationary distribution. In particular then, we find that $A_s^{(-\mu)}$ is distributed as an inverse gamma distribution with parameters $\mu$ and $\frac{1}{2}$. \qed
Appendix.

Lemma. If is a $\sigma-$algebra then following are equivalent:

1. $\mathcal{F}$ is countably generated as a $\sigma-$algebra
2. There are countably many real valued random variables $X_i$ so that $\sigma(X_i) = \mathcal{F}$
3. There exists a bounded real valued random variable $X$ so that $\sigma(X) = \mathcal{F}$

Proof. $1 \implies 2$ follows from taking characteristic functions. $2 \implies 1$ follows by taking inverse images of balls with rational endpoints. $1 \implies 3$ by taking a generating set $A_k$ and setting

$$X = \sum_{k=0}^{\infty} 2^{-k} 1_{A_k}$$

and observing that $\omega \in A_k$ if and only if the $k^{th}$ digit in the binary expansion of $X(\omega) = 1$. Finally $3 \implies 1$ by taking inverse images of balls with rational endpoints.

Lemma. If $\mathcal{F}_t$ is a separable filtration then there exists a bounded real valued random variable $L$ with $\mathcal{F}_\infty = \sigma(L)$.

Proof. For each $n$ $\mathcal{F}_n = \sigma(X_n)$ for some bounded real valued random variable $X_n$. Then since $\mathcal{F}_t$ is a filtration (so that for every $t$ there exists $n$ with $F_t \subseteq F_n$) it follows that $\sigma(X_n, n \in \mathbb{N}) = F_\infty$. Consequently there exists a bounded real valued random variable $X$ with $\mathcal{F}_\infty = \sigma(X)$.

References