MATH 421 DISCUSSION WEEK 1 LESSON PLAN

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Hours: Tu 1:00-2:00, Th 12:00-2:00

Course Philosophy

MATH 421 is an introduction to rigorous mathematics through the foundations of calculus. This means that although much of the material will be familiar to students, the method of presentation will likely not be. In terms of teaching, this means that we will try to abstract away as much of the structure we are familiar with as possible. Grading will be focused on completeness and correctness.

Discussion will be organized informally. Students should feel free to interrupt me, ask questions during class, and should ask for help if they need it. I believe in learning by example and so I will present example problems, then pose questions and give students time to work in small groups on them. Once there is some consensus within and among the groups, we will discuss the problems as a class. I will also consider solving some homework problems if given sufficient notice to prepare the questions.

The Field Axioms

In this class, we will be following an axiomatic approach. The typical format will be “definition, theorem, proof” so we should get started with practicing that style as soon as possible. We will begin with the ‘field axioms’, which are really more algebraic than analytic and which describe the basic arithmetic structure of real numbers:

- (P1) $a + (b + c) = (a + b) + c$ (associativity of addition)
- (P2) There exists a number 0 such that for all $a$ $a + 0 = a$ (existence of additive identity)
- (P3) For each $a$ there exists a number $-a$ such that $(-a) + a = a + (-a) = 0$ (existence of two-sided additive inverses)

Remark. Pause here to ask what kinds of groups we know with this property? Reals, rationals, complex numbers, integers. What kinds of operations fail each of these? Examples: subtraction, integration of products, exponentiation, etc.
Lemma. Now show that we can now prove if \( a + x = a \) then \( x = 0 \) (so 0 is unique). Show also that two-sided inverses are unique.

- (P4) \( a + b = b + a \)

Remark. Pause to ask about (P3) now and how I should have written it. Comment that sometimes an operation is defined which is not commutative, but that we typically do not call such an operation “addition”.

- (P5) \( a \cdot (b \cdot c) = (a \cdot b) \cdot c \)
- (P6) There exists a number 1 such that \( a \cdot 1 = a \) and \( 1 \neq 0 \)
- (P7) For each \( a \neq 0 \) there exists a number \( a^{-1} \) such that \( a \cdot a^{-1} = a^{-1} \cdot a = 1 \)

Notice that division is only defined through multiplication and subtraction is only defined through addition.

Remark. Of the examples we discussed before, which satisfy these properties as well? Notice that the integers do not, so if we prove things using these next three properties, they will not necessarily hold for the integers.

- (P8) \( a \cdot b = b \cdot a \)

Can we think of any examples

- (P9) \( a \cdot (b + c) = a \cdot b + a \cdot c \)

Remark. Why do I need the assumption \( 1 \neq 0 \) in (P6). Couldn’t we just prove that \( 1 \neq 0 \)? Isn’t it obvious? Use the example of the set \( a \) with \( a + a = a \) and \( a \cdot a = a \) 1 = a = 0 as a counterexample.

Problem. Show that if \( a \neq 0 \) and \( a \cdot b = a \cdot c \) then \( b = c \).

\[
b = 1 \cdot b = (a^{-1} \cdot a) \cdot b = a^{-1}(a \cdot b) = a^{-1}(a \cdot c) = (a^{-1} \cdot a) \cdot c = 1 \cdot c = c
\]

Problem. Show that \( a \cdot 0 = 0 \) for all \( a \).

\[
a \cdot 0 = a \cdot (0 + 0) = a \cdot 0 + a \cdot 0 \text{ so by our theorem, } a \cdot 0 = 0.
\]

Problem. Show that you cannot divide by zero, i.e. \( 0^{-1} \) cannot exist.

If it did, then \( 1 = 0^{-1} \cdot 0 = 0 \) by the previous problem, a contradiction.

Show as an example that \( a \cdot (−1) = −a \) and explain what the two sides of this equation are.

\[
0 = 0 \cdot a = a \cdot (1 + (−1)) = a \cdot 1 + a \cdot (−1) = a + a \cdot (−1)
\]

and by uniqueness, \( a \cdot (−1) = −a \).