This week we are going to work on inverse functions. I really want to highlight that generally \( f^{-1}(x) \neq \frac{1}{f(x)} \).

To start with, let’s discuss what an inverse actually is. The inverse image \( f^{-1}(y) \) is the set of all \( x \) with \( f(x) = y \). For example \( \sin^{-1}(0) = \{k\pi \text{ for } k \in \mathbb{Z}\} \).

Similarly, if \( f(x) = x^2 \) then \( f^{-1}(y) = \{\sqrt{y}, -\sqrt{y}\} \).

We say that a function \( f \) is invertible if \( f^{-1}(y) \) has only one element in it for each \( y \) (i.e. the map that takes \( y \) to \( f^{-1}(y) \) defines a function). Notice that this implies that \( \sin \) is not invertible. I only want to do a little with this, so let’s just recall one big theorem:

**Theorem.** If \( f \) is a continuous function on an interval then \( f \) is one to one if and only if \( f \) is strictly increasing or decreasing.

There were some questions about one of the examples that the professor did in class on the last day before break, so I wanted to go over that again first.

We would like to show the existence of a function satisfying

\[
[f(x)]^5 + f(x) + x = 0
\]

which we rewrite as

\[-[f(x)]^5 - f(x) = x.
\]

Looking at the problem this way suggests looking for the function \( f(x) \) which is the inverse function of \( g(y) = -y^5 - y \). Notice that if \( f(x) \) is the inverse function of \( g \) then \( g(f(x)) = x \), or \(-[f(x)]^5 - f(x) = x \). It therefore suffices to show that \( g \) is invertible. The theorem above gives us a nice test for invertibility: if we can show that \( g \) is strictly decreasing, then we are done.

\[
g'(y) = -5y^4 - 1 < 0
\]

for all \( y \). This shows that \( g \) is a strictly decreasing function. Therefore \( g \) is invertible and if we call \( g^{-1}(x) = f(x) \) we have found \( f \) with

\[-[f(x)]^5 - f(x) = x.
\]

We used the result that a strictly decreasing or increasing function is invertible above, but we can often actually solve for inverse functions algebraically.

**Example.** Find \( f^{-1}(y) \) for \( f(x) = (x - 1)^3 \). If \( y = (x - 1)^3 \) then \( y^{\frac{1}{3}} = x - 1 \) so \( x = y^{\frac{1}{3}} + 1 \). We have implicitly used the fact that \( f(x) = x^3 \) is invertible to prove this—this follows from the fact that \( f(x) \) is a strictly increasing function.
Example. If 

\[ f(x) = \begin{cases} 
  x & \text{x rational} \\
  -x & \text{x irrational} 
\end{cases} \]

then take \( y \) with \( f(x) \). Notice that \( f(x) \) is rational if and only if \( x \) is rational. Then suppose \( f(x) = y \) where \( y \) is irrational; we find that \( y = -x \) so \( x = -y \). Similarly for \( f(x) = y \) rational, we have \( x = y \). It follows then that

\[ f^{-1}(y) = \begin{cases} 
  y & \text{y rational} \\
  -y & \text{y irrational} 
\end{cases} \]

Example. Consider

\[ f(x) = \begin{cases} 
  -x^2 & x \geq 0 \\
  1 - x^3 & x < 0 
\end{cases} \]

We start by figuring out what the range of this thing is: it’s actually all of \( \mathbb{R} \). Notice that if \( f(x) \leq 0 \) then \( x \geq 0 \) and if \( f(x) > 0 \) then \( x < 0 \). Take \( y \in [0, \infty) \), then \( f^{-1}(y) = \sqrt{-y} \) and for \( y \in (-\infty, 0) \) \( f^{-1}(y) = -(y - 1)^{\frac{1}{3}} \).

Problem. Find \( f^{-1}(y) \) for \( f(x) = \frac{x}{1-x^2} \) \( -1 < x < 1 \).

If \( f(x) = 0 \) then \( x = 0 \). If \( f(x) = y > 0 \) then 

\[ y = \frac{x}{1-x^2} \implies \frac{1}{y} = \frac{1-x^2}{x} = \frac{1}{x} - x \]

so

\[ x^2 + x \frac{1}{y} - 1 = 0 \]

and we find that the inverse function is \( \frac{-1 \pm \sqrt{1+4y^2}}{2y} \) by the quadratic formula. Notice that I haven’t actually given the inverse function yet: we still need to figure out whether there should be a positive sign or a negative sign. We can determine that the correct function is \( \frac{-1+\sqrt{4y^2+1}}{2y} \) by observing that \( \frac{x}{1-x^2} \to \infty \) as \( x \to 1 \) while \( \frac{x}{1-x^2} \to -\infty \) as \( x \to -1 \).

Problem. On which intervals is \( f(x) = x^3 - 3x^2 \) one to one?

We first find where the derivative is positive or negative. \( f'(x) = 3x^2 - 6x = 3x(x-2) \) which is zero at 0 and 2. For \( x < 0 \), this is strictly positive; for \( 0 < x < 2 \) it is strictly negative; for \( x > 2 \) it is positive again. Since the function goes from increasing to decreasing and decreasing to increasing at 0 and 2 respectively, it is not 1–1 on any larger intervals.

Recall that we have the following theorem:
Theorem. Let \( f \) be a continuous one-to-one function defined on an interval and suppose that \( f \) is differentiable at \( f^{-1}(b) \) with derivative \( f'(f^{-1}(b)) \neq 0 \). Then \( f^{-1} \) is differentiable at \( b \) and

\[
(f^{-1})'(b) = \frac{1}{f'(f^{-1}(b))}
\]

Problem. Show that if \( f \) is a one-to-one function and \( f^{-1} \) is defined on the entire real line and has a derivative which is nowhere zero then \( f \) is differentiable.

Proof. Notice that \( f = (f^{-1})^{-1} \) so we may directly apply the previous theorem. \( \square \)