This is our last week, so today we are just going to do review problems. There are only three since I expect the first problem will take a while. It uses a lot of the ideas from the class.

**Problem.** Show that

\[
f(x) = \begin{cases} 
0 & x \leq 0 \\
e^{-\frac{1}{x^2}} & x > 0
\end{cases}
\]

is infinitely differentiable and that \( f^{(n)}(0) = 0 \) for each \( n \in \mathbb{N} \).

**Proof.** Clearly this function is differentiable at \( x \) if \( x \neq 0 \). Recall that the mean value theorem gives that for \( x > 0 \)

\[
\frac{f(x)}{x} = f'(\xi)
\]

for some \( \xi \in (0, x) \) and similarly if \( x < 0 \). To show that

\[
\lim_{x \to 0} \frac{f(x) - f(0)}{x - 0}
\]

exists, it suffices to show that

\[
\lim_{x \to 0^+} f'(x)
\]

exists. Clearly

\[
\lim_{x \to 0^-} f(x) = 0
\]

so it suffices to show that

\[
\lim_{x \to 0^+} f'(x)
\]

exists. Now, for \( x > 0 \)

\[
f'(x) = 2 \frac{2}{x^3} e^{-\frac{1}{x^2}}
\]

and we may compute \( \lim_{x \to 0} f'(x) \) by observing that

\[
\lim_{x \to 0^+} \frac{e^{-\frac{1}{x^2}}}{x^3}
= \lim_{x \to \infty} \frac{x^3}{e^{x^2}} = 0
\]
The last limit can be computed with L’Hopital’s rule or by using Taylor’s theorem on $e^{x^2}$. We can now show by induction that $f^{(n)}(x)$ is of the form

$$f^{(n)}(x) = \begin{cases} 0 & x \leq 0 \\ P_n\left(\frac{1}{x}\right)e^{\frac{-1}{x^2}} & x > 0 \end{cases}$$

for a polynomial $P_n(x)$. The base case is done above. Notice that if $P_n(x)$ is a polynomial then

$$\frac{d}{dx} P_n\left(\frac{1}{x}\right)e^{\frac{-1}{x^2}} = P_n'\left(\frac{1}{x}\right)\frac{-1}{x^2}e^{\frac{-1}{x^2}} + 2\frac{1}{x^3}P_n\left(\frac{1}{x}\right)e^{\frac{-1}{x^2}}$$

which is of the desired form. It only remains to be shown that

$$\lim_{x \to 0^+} P_n\left(\frac{1}{x}\right)e^{\frac{-1}{x^2}} = 0$$

It suffices to show that $\lim_{x \to 0^+} \frac{1}{x^2}e^{\frac{-1}{x^2}} = 0$, which can be shown in the same way as above.

□

**Problem.** Show that the differential equation $\frac{d}{dx} f'(x) = cf(x)$ with $f(0) = 1$ has a unique solution.

(Hint: we did something similar in the chapter on trig functions with the equation $\frac{d^2}{dx^2} f(x) = -f(x)$—this is easier).

**Proof.** Whenever you do a problem like this, the first thing to try to do is find something that should be constant. First observe that $f(x) = e^{cx}$ is a solution to the differential equation. Now suppose that $g(x)$ is another function with $g'(x) = cg(x)$ and $g(0) = 1$. Then

$$\frac{d}{dx} e^{-cx}g(x) = -ce^{-cx}g(x) + g'(x)e^{-cx}$$

$$= -e^{-cx}g'(x) + g'(x)e^{-cx} = 0$$

since $cg(x) = g'(x)$ by hypothesis. Thus $e^{-cx}g(x) = C$ for some constant $C$. Evaluating at 0 gives that $C = 1$ so $g(x) = e^{cx}$.

□

**Problem.** Suppose that $f(x)$ is an increasing function on $[a, b]$. Show that $f(x)$ is integrable on $[a, b]$.

**Proof.** First observe that $f(a) \leq f(x) \leq f(b)$ for any $x \in [a, b]$ by hypothesis, so that $f(x)$ is bounded. Now, choose a uniform partition $P_n$ of $[a, b]$ into segments of length $\frac{b-a}{n}$ and call $x_k = a + k\frac{b-a}{n}$ where $0 \leq k \leq n$. Then

$$U(f, P) - L(f, P) = \sum_{k=0}^{n-1} [f(x_{k+1}) - f(x_k)] \frac{b-a}{n}$$

$$= \frac{b-a}{n} \sum_{k=0}^{n-1} f(x_{k+1}) - f(x_k)$$
\[ \frac{(b - a)}{n} (f(b) - f(a)) \]

and taking \( n \to \infty \) we may make this less than \( \epsilon \) for any \( \epsilon \) so \( f \) is integrable. \( \Box \)