**Math 421 Lesson Plan Week 8**

**Chris Janjigian**

This week we are going to talk about uniformly continuous functions and differentiability of functions. I will start off with some examples with proofs that functions are not uniformly continuous. Let’s start off by recalling the definition of uniform continuity:

**Definition.** A function \( f(x) \) is said to be uniformly continuous on a set \( A \) if for all \( x, y \in A \) and \( \varepsilon > 0 \) there exists a \( \delta \) (which does not depend on \( x \) and \( y \)—this is the difference between continuity and uniform continuity) so that for \( |x - y| < \delta \) we have \( |f(x) - f(y)| < \varepsilon \).

As a comment, on open intervals \((a, b)\) uniform continuity of \( f(x) \) is equivalent to the existence of a function \( g(x) \) so that \( f(x) = g(x) \) on \((a, b)\) and \( g(x) \) is continuous on \([a, b]\). I mention this in part to help with the homework and in part to connect this concept with something we did before.

**Example.** \( f(x) = \frac{1}{x} \) is not uniformly continuous on \((0, 1)\)

**Proof.** Suppose that \( f(x) \) was uniformly continuous on \((0, 1)\) and fix \( \varepsilon = 1 \). Then there exists \( \delta > 0 \) such that \( |f(x) - f(y)| < 1 \) if \( |x - y| < \delta \). We are now going to produce specific values of \( x \) and \( y \) that give us a contradiction. We know that there exists an \( N \) such that \( \frac{1}{N} < \delta \), but since we also have \( 0 < \frac{1}{N} - \frac{1}{N+1} < \frac{1}{N} < \delta \). In particular then we must have

\[
|f\left(\frac{1}{N}\right) - f\left(\frac{1}{N+1}\right)| < 1
\]

But \( f\left(\frac{1}{N}\right) = N \) while \( f(N+1) = N+1 \), so \( |f\left(\frac{1}{N}\right) - f\left(\frac{1}{N+1}\right)| = 1 \), a contradiction. \(\square\)

**Exercise.** (Homework problem) Give an example of a bounded function on \((0, 1]\) which is not uniformly continuous. (Hint: Think about previous examples from discussion and the comment above)

**Proof.** Take \( f(x) = \sin\left(\frac{1}{x}\right) \) and suppose that \( f(x) \) is uniformly continuous on \((0, 1]\). Take \( \varepsilon = 1 \); then there is a \( \delta > 0 \) such that for \( |x - y| < \delta \) we have \( |f(x) - f(y)| < 1 \). But we know that since \( \frac{1}{2 + 2\pi n} < \frac{1}{2\pi n} < \frac{1}{n} \), there exists an \( N \in \mathbb{N} \) with \( 0 < \frac{1}{2 + 2\pi N} - \frac{1}{2\pi N} < \frac{1}{2 + 2\pi N} < \frac{1}{n} \), there exists an \( N \in \mathbb{N} \) with \( 0 < \frac{1}{\frac{1}{2} + 2\pi N} - \frac{1}{\frac{1}{2} + 2\pi N} < \frac{1}{\frac{1}{2} + 2\pi N} < \delta \). Then we must have

\[
|f\left(\frac{1}{2 + 2\pi N}\right) - f\left(\frac{\pi}{2} \right) - 2\pi N) - \frac{1}{2\pi N} | < 1,
\]
but we know that $f\left(\frac{1}{2} + 2\pi N\right) = \sin(\frac{\pi}{2} + 2\pi N) = 1$ and $f\left(\frac{1}{3\pi} + 2\pi N\right) = \sin(\frac{3\pi}{2} + 2\pi N) = -1$, so $|f\left(\frac{1}{2} + 2\pi N\right) - f\left(\frac{1}{3\pi} + 2\pi N\right)| = 2$ a contradiction. □

Uniform continuity behaves a little differently at infinity. The typical problems that arise with uniform continuity are rapid growth as in the first example or oscillations as in the second example. When looking for functions that are continuous but not uniformly continuous, you typically want to look for one of these two things. A useful fact to know, which is almost identical to one of the questions on the last exam is the following:

**Exercise.** Suppose that $f(x)$ satisfies $|f(x) - f(y)| \leq c|x - y|$ for all $x, y \in A$. Then $f(x)$ is uniformly continuous on $A$.

**Proof.** Fix $\epsilon > 0$ and set $\delta = \frac{\epsilon}{c}$. Then for $|x - y| < \delta$ we have $|f(x) - f(y)| \leq c|x - y| < \epsilon$. □

Uniform continuity can be tricky sometimes, so when I approach problems about uniform continuity, I typically use the previous result to rule out examples. The way you usually use this is with the mean value theorem (which we haven’t proven yet, so you can’t use in proofs!). We know that $\left|\frac{f(x) - f(y)}{x - y}\right| = |f'(z)|$ for some $z$ with $x < z < y$ (assuming $x < y$). Therefore if the derivative of a differentiable function is bounded, then it is uniformly continuous. As an example of how to use this (to emphasize, this is to give you a way to think about problems, you cannot use this on proofs since we have not proven the mean value theorem) consider the following

**Example.** Assuming the mean value theorem, show that $x^{\frac{1}{2}}$ is uniformly continuous on $[1, \infty)$.

**Proof.** For $f(x) = x^{\frac{1}{2}}$ we have $|f'(x)| = \left|\frac{1}{2x^{\frac{1}{2}}}\right| \leq \frac{1}{2}$ on $[1, \infty)$. Then since $|f'(x)| \leq \frac{1}{2}$ the previous result shows that $f$ is uniformly continuous on $[1, \infty)$. □

Although you cannot use this in proofs yet, it does tell you how to approach one of the problems on the homework. I should also mention that a good way to find examples where you can expect uniform continuity to fail is to find values of $a$ with $|f'(x)| \to \infty$ as $x \to a$. This does not always work (these concepts are not equivalent) but it is often a good place to start.

**Exercise.** (from the homework) Find two functions $f(x)$ and $g(x)$ defined on $\mathbb{R}$ and both uniformly continuous so that $f(x)g(x)$ is not uniformly continuous.

**Proof.** $f(x) = g(x) = x^{2}$. As a hint for how to do the proof that $x^{2}$ is not uniformly continuous, follow the examples above and take $x = N + \frac{1}{N}$ and $y = N$ for $N$ large. □

Now we can move on to differentiability of functions properly. Again, let us recall the definition of differentiability:

**Definition.** A function $f(x)$ is differentiable at $x = a$ if there exists a limit $f'(a)$ such that

$$
\lim_{x \to a} \frac{f(x) - f(a)}{x - a} = f'(a).
$$

If such a limit exists, then $f(x)$ is differentiable at $x = a$, and $f'(a)$ is its derivative. If $f(x)$ is differentiable at every point of an open interval $I$, then $f(x)$ is said to be differentiable on $I$.

The derivative of a differentiable function $f(x)$ at a point $x = a$ is given by

$$
f'(a) = \lim_{h \to 0} \frac{f(a + h) - f(a)}{h}.
$$

The process of finding the derivative of a function is called differentiation. The function $f'(x)$ is called the derivative of $f(x)$.

Uniform continuity is important in proving differentiability. If a function is uniformly continuous, then its derivative exists almost everywhere. This is a consequence of the mean value theorem.

**Theorem.** If $f(x)$ is uniformly continuous on $[a, b]$, then $f(x)$ is continuously differentiable almost everywhere on $[a, b]$.

**Proof.** Let $f(x)$ be uniformly continuous on $[a, b]$. Then for every $\epsilon > 0$, there exists $\delta > 0$ such that $|x - y| < \delta$ implies $|f(x) - f(y)| < \epsilon$. Let $x_0 \in [a, b]$ be a fixed point. Then for $x, y \in [a, b]$ with $|x - y| < \delta$, we have

$$
|f'(x_0)| = \left|\lim_{h \to 0} \frac{f(x_0 + h) - f(x_0)}{h}\right| \leq \lim_{h \to 0} \frac{|f(x_0 + h) - f(x_0)|}{|h|} < \frac{\epsilon}{\delta}.
$$

Thus $f'(x_0)$ exists and is bounded. Therefore $f(x)$ is differentiable at $x_0$.

Now we can move on to differentiability of functions properly. Again, let us recall the definition of differentiability:
Definition. A function $f$ is said to be differentiable at $a$ if

$$\lim_{x \to a} \frac{f(x) - f(a)}{x - a} = L$$

exists, in which case we call $L = f'(a)$.

Let's do an explicit computation for a function that looks a little different from what we are used to:

Example. Let

$$f(x) = \begin{cases} x^2 & \text{if } x \text{ is rational} \\ 0 & \text{if } x \text{ is irrational} \end{cases}$$

Then $f(x)$ is differentiable at zero and $f'(0) = 0$.

Proof. Take $0 < |x| < \epsilon$. Then if $x$ is irrational

$$\left| \frac{f(x) - f(0)}{x - 0} - 0 \right| = \frac{0}{x} = 0$$

If $x$ is rational, then

$$\left| \frac{f(x) - f(0)}{x - 0} - 0 \right| = \left| \frac{x^2}{x} \right| = |x| < \epsilon$$

which implies that $f$ is differentiable at $0$ and $f'(0) = 0$. □

You are asked on your homework to prove a generalization of this result. The case of $x$ rational above is essentially the proof of that, so make sure you understand it.

Problem. Suppose that $f(x)$ has the property for all real numbers $x, y$ we have $|f(x) - f(y)| \leq |x - y|^2$. Show that $f$ is differentiable at zero and that $f'(0) = 0$. (In fact, it is not hard to prove that the only such functions are the constant functions).

Proof. Notice that $\frac{|f(x) - f(0)|}{|x|} \leq |x|$ so taking $x \to 0$ gives that $\lim_{x \to 0} \frac{f(x) - f(0)}{x - 0}$ exists and equals zero. □

Problem. Suppose that $f(x)$ is a function with the property that for $x, y \in A$

$$|f(x) - f(y)| \leq c|x - y|^\alpha$$

for $c > 0$ and $\alpha > 0$ then $f$ is uniformly continuous on $A$.

Proof. Fix $\epsilon > 0$ and take $\delta = \left( \frac{\epsilon}{c} \right)^{\frac{1}{\alpha}}$ (we know that these numbers exist now, so this is OK). Then for $|x - y| < \delta$ we have

$$|f(x) - f(y)| \leq c|x - y|^\alpha < c \left( \left( \frac{\epsilon}{c} \right)^{\frac{1}{\alpha}} \right)^\alpha = \epsilon$$

□
**Problem.** Show that for

\[ f(x) = \begin{cases} 
  x & x \text{ rational} \\
  0 & x \text{ irrational}
\end{cases} \]

\( f(x) \) is not differentiable at 0.

**Proof.** Suppose that \( f(x) \) was differentiable at 0, so that \( \lim_{x \to 0} \frac{f(x) - f(0)}{x - 0} = L \) exists. Then taking \( x_n \to 0 \) along a sequence of rational numbers, we find that

\[ \frac{f(x_n) - f(0)}{x_n - 0} = \frac{x_n}{x_n} = 1 \to 1 \]

so \( L = 1 \). Conversely, taking \( x_n \to 0 \) along irrational numbers, we find that

\[ \frac{f(x_n) - f(0)}{x_n - 0} = \frac{0}{x_n} = 0 \to 0 \]

so \( L = 0 \), a contradiction. \( \square \)