We are going to continue with our discussion of differentiability from last week. I am going to start with an example you should keep in mind when trying to do proofs in this section.

**Theorem.** There exists a differentiable function with a discontinuous derivative. *(Also, the intermediate value property does not imply continuity)*

**Proof.** Define

\[
g(x) = \begin{cases} 
0 & x = 0 \\
x^2 \sin(\frac{1}{x}) & x \neq 0
\end{cases}
\]

Notice that \(|g(x)| \leq x^2\) so by one of your homework problems (which we basically proved in section last week) \(g\) is differentiable at 0 and \(g'(0) = 0\). For \(x \neq 0\) \(g(x) = x^2 \sin(\frac{1}{x})\) which is a product of differentiable functions and therefore is differentiable. Then

\[
g'(x) = \begin{cases} 
0 & x = 0 \\
2x \sin(\frac{1}{x}) - \cos(\frac{1}{x}) & x \neq 0
\end{cases}
\]

Notice that \(\lim_{x \to 0} g'(x)\) does not exist, since \(\lim_{x \to 0} \cos(\frac{1}{x})\) does not exist. In particular then, \(g'\) is not continuous at zero. \(\square\)

It turns out that derivatives always satisfy the intermediate value property and if you check you can see that this also gives an example of a function which satisfies the intermediate value property, but is discontinuous.

After giving that example, I would like to move on to some more consequences of the mean value theorem, which I restate:

**Theorem.** *(Mean Value Theorem)* Suppose that \(f\) is continuous on \([a, b]\) and differentiable on \((a, b)\). Then there exists \(c \in (a, b)\) with

\[
\frac{f(b) - f(a)}{b - a} = f'(c)
\]

Intuitively what this is saying is that if we think of \(f\) as a position function, then a moving object always attains its average velocity over any period of time (the left hand side is the total distance travelled over the total time travelled, while the right hand side is an instantaneous velocity).

For the next result, we are going to use a consequence of the mean value theorem proven in class.
**Proposition.** Suppose that $f$ is a differentiable function on an interval and $f'(x) = 0$ for all $x$ in that interval. Then $f$ is constant.

**Proof.** Fix $x < y$; then there exists $z$ with $x < z < y$ and

$$\frac{f(y) - f(x)}{y - x} = f'(z) = 0$$

Then $f(x) = f(y)$. Since $x$ and $y$ were arbitrary, it follows that $f(x) = f(y)$ for all $x$ and $y$ in the interval. \(\square\)

Last week, we discussed the Lipschitz property that if $|f(x) - f(y)| \leq c|x - y|$ it follows that $f$ is uniformly continuous. It turns out that in some sense this is the limit of how much information we can get by considering functions with $|f(x) - f(y)| \leq C|x - y|^\alpha$:

**Problem.** Suppose that $f$ is defined on an interval and $|f(x) - f(y)| \leq c|x - y|^\alpha$ for some $c > 0$ and $\alpha > 1$. Then $f$ is a constant function.

**Proof.** Notice that for any $x < y$ we have

$$0 \leq |\frac{f(y) - f(x)}{y - x}| \leq c|x - y|^{1-\alpha}$$

Taking $y \to x$ and applying the squeeze theorem gives that $f'(x)$ exists and is zero. This holds for any $x$ and any $y$ so $f$ is constant. \(\square\)

**Problem.** Suppose that $f$ is Lipschitz with constant $C$ (this means $|f(x) - f(y)| \leq C|x - y|$) and differentiable. Show that

$$|f'(x)| \leq C$$

**Proof.** We have

$$|f(x) - f(y)| \leq C|x - y|$$

so

$$|\frac{f(x) - f(y)}{x - y}| \leq C$$

The limit as $y \to x$ on the left exists and equals $|f'(x)|$ so we conclude that $|f'(x)| \leq C$. \(\square\)

Another very important property following from the mean value theorem is the result that functions with strictly positive derivatives are strictly increasing:

**Proposition.** Suppose that $f$ is a differentiable function on an interval and $f'(x) > 0$ for all $x$ in that interval. Then $f$ is increasing.

**Proof.** Take $x < y$ in our interval; then there is $z$ with $x < z < y$ and

$$\frac{f(y) - f(x)}{y - x} = f'(z) > 0$$
so
\[ f(y) - f(x) > 0 \]
and thus \( f(y) > f(x) \). \( \square \)

**Problem.** Show that if \( g(x) \) is a differentiable function with \( |g'(x)| \leq C \) then there exists \( \epsilon > 0 \) such that \( f(x) = x + \epsilon g(x) \) is an increasing function.

**Proof.** Take \( \epsilon = \frac{1}{2C} \) for example. Then
\[
f'(x) = 1 + \epsilon g(x) \geq 1 - \frac{1}{2C} \cdot C = \frac{1}{2}
\]
\( \square \)

Recall that \( \frac{d}{dx} \tan(x) = \sec^2(x) \) for the next problem:

**Problem.** Show that \( \tan(x) > x \) for \( 0 < x < \frac{\pi}{2} \).

**Proof.** Take \( x > 0 \). Then by the mean value theorem there is \( y \) with \( 0 < y < x \) and
\[
\frac{\tan(x) - \tan(0)}{x - 0} = \sec^2(y) > 1
\]
and therefore \( \tan(x) > x \). \( \square \)