REVIEW SOLUTIONS

Problem. (1)
(i.) Consider $x = 1$, $y = 2$, and $z = -1$. Then $x < y$ but $xz > yz$.
(ii.) Take $x = -1$ and $y = 0$. Then $x < y$ but $|x| = |y + 1|$.
(iii.) Take $x = 0$
(iv.) Take $f(x) = x^2 : \mathbb{R} \to \mathbb{R}^+$ and $g(x) = \sqrt{x} : \mathbb{R}^+ \to \mathbb{R}$. Then (recall that $f \circ g$ is a function from $\mathbb{R}^+$ to $\mathbb{R}^+$)
\[ f \circ g(x) = \sqrt{x^2} = x \]
but (recall that $g \circ f$ is a function from $\mathbb{R}$ to $\mathbb{R}$)
\[ g \circ f(x) = \sqrt{x^2} = |x| \geq 0 \]

Problem. (2) $-1.5 \leq x \leq 1.5$

Problem. (3) Suppose that $\sqrt{10} = \frac{a}{b}$ with $a$ and $b$ coprime. Then $10 = 5 \cdot 2 = \frac{a^2}{b^2}$ so $10b^2 = a^2$. Therefore $2|a$. Write $a = 2n$; then $10 = \frac{4n^2}{b^2}$ so $5b^2 = 2n^2$ and therefore since 2 does not divide 5 we know that $2|b$, contradicting that $a$ and $b$ are coprime.

Problem. (4) Base case $n = 1 : 1 = 1^2$
Induction step: Suppose that $\sum_{k=1}^{n} 2(k - 1) + 1 = n^2$
Then
\[ \sum_{k=1}^{n+1} 2(k - 1) + 1 = \sum_{k=1}^{n} 2(k - 1) + 1 + 2n + 1 = n^2 + 2n + 1 = (n + 1)^2 \]

Problem. (5) Let $h > -1$. We will show that $(1 + h)^n \geq 1 + nh$ by induction.
Base case $n = 1$: $1 + h = 1 + h$.
Induction Step: Assume that $(1 + h)^n \geq 1 + nh$
\[ (1 + h)^{n+1} = (1 + h)^n (1 + h) \geq (1 + nh)(1 + h) = 1 + (n + 1)h + nh^2 \geq 1 + (n + 1)h \]
since $h^2 \geq 0$ and $n > 0$. 

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Problem. (6) We have

\[ f(x) = \frac{(x - x^3)\sqrt{5 - x}}{(x + 2)\sqrt{x - 3}} \]

This function will be well-defined if \(5 - x \geq 0\), \(x + 2 \neq 0\) and \(x - 3 > 0\). These conditions combine to \(3 < x \leq 5\).

Problem. (7) Let \(\epsilon > 0\) and set \(\delta = \min\left(\frac{3}{1+\epsilon} - 3, |\frac{3}{1+\epsilon} - 3|\right)\). Then for

\[ 0 < |x - 2| < \delta \]

we have

\[ \frac{3}{1+\epsilon} - 3 < x - 2 < \frac{3}{1-\epsilon} - 3 \]

Then

\[ \frac{3}{1+\epsilon} < x + 1 < \frac{3}{1-\epsilon} \]

and therefore

\[ \epsilon < \frac{3}{x + 1} - 1 < -\epsilon \]

which is if and only if

\[ \frac{3}{x + 1} - 1 < \epsilon \]

Problem. (8) Suppose that

\[ \lim_{x \to a} \frac{1}{x - a} = L \]

exists and pick \(\epsilon = 1\). Then there is a \(\delta > 0\) so that for \(|x - a| < \delta\) we have \(|\frac{1}{x-a} - L| < 1\). Notice that this implies that

\[ |\frac{1}{x-a}| = |\frac{1}{x-a} - L + L| \leq 1 + |L|. \]

There is an \(N_1 \in \mathbb{N}\) so that \(N_1 > 1 + |L|\) and there is \(N_2 \in \mathbb{N}\) so that \(\frac{1}{N_2} < \delta\). Set \(N = \max(N_1, N_2)\). Then for \(x = a + \frac{1}{N}\) we have

\[ |(a + \frac{1}{N}) - a| = \frac{1}{N} < \delta \]

but

\[ \frac{1}{a + \frac{1}{N} - a} = \frac{1}{N} = N > 1 + |L| \]

a contradiction.

Problem. (9) Remember that in class we discussed the function

\[ f(x) = \begin{cases} x & \text{if } x \text{ is rational} \\ -x & \text{if } x \text{ is irrational} \end{cases} \]
which is continuous only at zero. We can turn this into a function continuous only at $a$ by translation:

$$f_a(x) = \begin{cases} 
  x - a & \text{if } x - a \text{ is rational} \\
  a - x & \text{if } x - a \text{ is irrational}
\end{cases}$$

**Problem.** (10) $f(x) = \max(f(x), 0) - \max(-f(x), 0)$ and both of these functions are non-negative and continuous.

There is something we need to prove here though: if $f(x)$ and $g(x)$ are continuous, then $\max(f(x), g(x))$ is continuous. This can be done with $\epsilon - \delta$ from the definition of a maximum but follows very quickly from the algebraic fact that

$$\max(a, b) = \frac{a + b + |a - b|}{2}$$

Therefore we have that

$$\max(f(x), g(x)) = \frac{f(x) + g(x) + |f(x) - g(x)|}{2}$$

Since differences of continuous functions are continuous, we find that $f(x) - g(x)$ is continuous. Because $|x|$ is continuous and compositions of continuous functions are continuous it follows that $|f(x) - g(x)|$ is continuous. Moreover, then since $f(x), g(x)$ and $|f(x) - g(x)|$ are continuous we know that $f(x) + g(x) + |f(x) - g(x)|$ since sums of continuous functions are continuous. Finally, since $\frac{5}{2}$ is continuous and compositions of continuous functions are continuous, we find that $\max(f(x), g(x)) = \frac{f(x)+g(x)+|f(x)−g(x)|}{2}$ is continuous.