Problem. (12.9) Notice that if $f$ is one to one and $f^{-1}$ has a derivative which is nowhere zero, since $f = (f^{-1})^{-1}$, we may apply theorem five from this chapter to obtain the result.

Problem. (12.27). Suppose that $f(x) > 0$ for all $x$ and that $f(x)$ is decreasing. If $f(x) > c > 0$ for all $x$, we are done, so we may without loss of generality assume that $f(x) \to 0$. Then by the intermediate value theorem for each $n$ there exists $x_n$ so that $f(x_n) = \frac{1}{n}$ and notice that for $x > x_n$ we have $f(x) < \frac{1}{n}$. On the interval $[x_n, x_{n+1}]$ we may linearly interpolate between $\frac{1}{n+1}$ and $\frac{1}{n+2}$: a formula for the the function that does this is in a fixed interval $[x_n, x_{n+1}]$ is

$$(1 - \frac{x - x_n}{x_{n+1} - x_n}) \frac{1}{(n + 2)^2} + (\frac{x - x_n}{x_{n+1} - x_n}) \frac{1}{(n + 3)^2}.$$ 

Let $g(x)$ be defined as above for all $n$. Notice that $g(x)$ is continuous with $g(x) > 0$ by construction and moreover that $\frac{1}{(n+3)^2} \leq g(x) \leq \frac{1}{(n+2)^2} < \frac{1}{n+1} \leq f(x) \leq \frac{1}{n}$ for $x \in [x_n, x_{n+1}]$. Thus for all $x$ we have $f(x) \geq g(x) > 0$. Further, for $x \in [x_n, x_{n+1}]$

$|\frac{g(x)}{f(x)}| \leq \frac{n+1}{(n+2)^2} \to 0$ as $n \to \infty$. Since taking $x \to \infty$ also takes $n \to \infty$, we are done.

Problem. (13.13) a.) Suppose that $f$ is integrable on $[a, b]$ and $f(x) \geq 0$. Then $\inf_{x \in [a, b]} f(x) \geq 0$, so if we let $P$ be the partition $\{a, b\}$ then $\int_a^b f \geq L(f, P) = \inf_{x \in [a, b]} f(x) (b - a) \geq 0$

b.) Now suppose that $f \geq g$. By the previous result $\int_a^b f - g \geq 0$ so $\int_a^b f \geq \int_a^b g$.

Problem. (13.26) a.) Suppose that $f$ is integrable on $[a, b]$ and fix $\epsilon > 0$. Choose a partition $P$ so that $U(f, P) - L(f, P) < \epsilon$ so that $U(f, P) - \int_a^b f < \epsilon$ and similarly for $L(f, P)$.

Since $U(f, P) = \sum_{i=1}^n M_i(t_{i+1} - t_i)$ we may define a step function $s_2(x) = \sum_{i=1}^n M_i 1_{[t_i, t_{i+1}]}(x)$ where

$$1_{[t_i, t_{i+1}]}(x) = \begin{cases} 1 & x \in [t_i, t_{i+1}] \\ 0 & \text{otherwise} \end{cases}$$

since the partition is disjoint $\int_a^b s_2(x) = U(f, P)$. A similar definition works for $s_1$. 

1
b.) Suppose that for all \( \epsilon > 0 \) there are step functions \( s_1 \leq f \) and \( s_2 \geq f \) with
\[
\int_a^b s_2 - s_1 < \epsilon
\]
Notice that any step function induces a partition by considering the set of points where it is discontinuous. We may without loss of generality assume that \( s_1 \) and \( s_2 \) induce the same partition \( P \). Fix a pair \( t_i, t_{i+1} \in P \). Since \( s_2 \) is constant on \([t_i, t_{i+1}]\) and \( s_2(x) \geq f(x) \) we know that \( \sup_{x \in [t_i, t_{i+1}]} f(x) \leq s_2(x) \). Since this holds independent of \( i \), it follows that
\[
\int_a^b s_2 \geq U(f, P)
\]
a similar argument gives that
\[
\int_a^b s_1 \leq L(f, P)
\]
therefore
\[
U(f, P) - L(f, P) \leq \int_a^b s_2 - s_1 < \epsilon
\]
so \( f \) is integrable.

c.) Let
\[
f(x) = \begin{cases} 
\frac{1}{q} & x = \frac{p}{q} \text{ in rational reduced form} \\
0 & \text{otherwise}
\end{cases}
\]
then \( \int_0^1 f(x)\,dx = 0 \) and for any partition \( P \) of \([0,1]\) we have \( L(f, P) = 0 \).
To prove that \( \int_0^1 f(x)\,dx = 0 \) fix \( \epsilon > 0 \) and notice that there is a \( Q \) for which \( \frac{1}{Q} < \epsilon \). Since there are only finitely many pairs \( (p,q) \) so that \( \frac{p}{q} \) is in reduced form with \( q < Q \) there are only finitely many rationals \( x_i \) with \( f(x_i) > \epsilon \). Let \( g(x) = f(x) \) except at these points and be zero at each of these \( x_i \). Then since these two differ at only finitely many points we have
\[
\int_0^1 g = \int_0^1 f
\]
and since \( |g(x)| < \epsilon \) for all \( x \) we have
\[
|\int_0^1 f| = |\int_0^1 g| \leq \int_0^1 |g| < \epsilon
\]
and since \( \epsilon \) is arbitrary, \( \int_0^1 f = 0 \).

**Problem.** 4.) ii.) Define \( f(x) = \int_1^x \cos(\cos(t))\,dt \) and observe that \( f'(x) = \cos(\cos(x)) > 0 \) since \(-1 \leq \cos(x) \leq 1 \) and since \( \cos(y) > 0 \) for \( y \in [-1,1] \) it
follows that $f$ is invertible and $(f^{-1})'(x) = \frac{1}{f'(f^{-1}(x))}$. Notice that $f^{-1}(0) = 1$ so we need

$$(f^{-1})'(0) = \frac{1}{f'(1)} = \frac{1}{\cos(\cos(1))}$$

**Problem.** (14.14) Suppose that $f$ is differentiable and one to one with a nowhere zero derivative and that $F' = f$. Observe that we have

$$\int_a^x f^{-1}(t)dt = xf^{-1}(x) - af^{-1}(a) - \int_{f^{-1}(a)}^{f^{-1}(x)} f(t)dt$$

$$= xf^{-1}(x) - af^{-1}(a) - F(f^{-1}(x)) + F(f^{-1}(a))$$

Since $F(f^{-1}(a)) - af^{-1}(a)$ is a constant, it follows that

$$\frac{d}{dx} xf^{-1}(x) - F(f^{-1}(x)) = f^{-1}(x)$$

**Problem.** (14.21) Recall that $f(x) - f(0) = \int_0^x f'(t)dt$ so

$$|f(x)| = \left| \int_0^x f'(t)dt \right|$$

$$\leq \int_0^x |f'(t)|dt$$

$$\leq \int_0^1 |f'(t)|dt$$

$$\leq \sqrt{\int_0^1 (f'(t))^2dt} \sqrt{\int_0^1 1dt}$$

$$= \sqrt{\int_0^1 (f'(t))^2dt}$$

**Problem.** (15.8) a.) Observe that

$$A \sin(x + B) = A[\sin(x) \cos(B) + \sin(B) \cos(x)]$$

$$= A \cos(B) \sin(x) + A \sin(B) \cos(x)$$

b.) Similarly to solve

$$a \sin(x) + b \cos(x) = A \sin(x + B)$$

we need to solve $A \cos(B) = a$ and $A \sin(B) = b$, which we can do as follows: set

$$B = \arccos\left(\frac{a}{A}\right)$$

so that

$$A = \frac{b}{\sin(B)} = \frac{b}{\sin(\arccos\left(\frac{a}{A}\right))}$$
The solution to this is $A = \pm \sqrt{a^2 + b^2}$, $B = \arccos\left(\frac{a}{\sqrt{a^2 + b^2}}\right)$ depending on the sign of $b$.

c.

$$f(x) = \sqrt{3} \sin(x) + \cos(x)$$

$$= 2 \sin\left(x + \frac{\pi}{6}\right)$$

**Problem.** (15.27) a.) Let $x$ be the angle in radians of the triangle pictured in the text, $0 < x < \frac{\pi}{4}$ and notice that $\frac{x}{2}$ is the area of the shaded region. The area of the triangle $OAB$ is $\frac{1}{2} \text{(base)} \times \text{(height)} = \frac{1}{2} (1) (\sin(x))$ so we have $\frac{\sin(x)}{2} < \frac{x}{2}$. If we drop a perpendicular from $A$ to $OB$ then we construct a triangle which is similar to $OCB$ with height $\sin(x)$ and base $\cos(x)$. Then $\frac{\sin(x)}{\cos(x)} = \frac{|CB|}{1}$ and we have our second inequality $\frac{x}{2} < \frac{\sin(x)}{2 \cos(x)}$.

b.) We may multiply through by $\frac{2}{\sin(x)}$ to find that

$$1 < \frac{x}{\sin(x)} < \frac{1}{\cos(x)}$$

and therefore

$$\cos(x) < \frac{\sin(x)}{x} < 1$$

thus by the squeeze theorem

$$\lim_{x \to 0^+} \frac{\sin(x)}{x} = 1$$

Since $\frac{\sin(x)}{x}$ is an even function, this implies that

$$\lim_{x \to 0} \frac{\sin(x)}{x} = 1$$

c.) Notice that $1 - \cos(x) = 2 \sin^2\left(\frac{x}{2}\right)$ so that

$$\lim_{x \to 0} \frac{1 - \cos(x)}{x} = \lim_{x \to 0} \frac{2 \sin^2\left(\frac{x}{2}\right)}{x}$$

if the second limit exists. We may change variables $y = \frac{x}{2}$ so that this is (if the following limits exist)

$$= \lim_{y \to 0} \frac{\sin^2(y)}{y} = \lim_{y \to 0} \sin(y) \lim_{y \to 0} \frac{\sin(y)}{y} = 0 \cdot 1 = 0$$
d.) Notice that
\[
\frac{\sin(x+h) - \sin(x)}{h} = \frac{\sin(x)\cos(h) + \sin(h)\cos(x) - \sin(x)}{h}
\]
\[
= -\sin(x)\frac{1 - \cos(h)}{h} + \cos(x)\frac{\sin(h)}{h}
\]
Since the limit as \( h \to 0 \) of both terms on the right exists, we may take
\[
\lim_{h \to 0} \frac{\sin(x+h) - \sin(x)}{h}
\]
and using the previous results, we find that this is \( \cos(x) \).

**Problem.** (20.13)
Define
\[
f(x) = \begin{cases} 
\frac{e^{x-1}}{x} & x \neq 0 \\
1 & x = 0
\end{cases}
\]
Notice that if \( g(x) = e^x \) then \( e^x - 1 = \sum_{i=1}^{n} \frac{x^i}{i!} + R_{n,0,g}(x) \) where \( R_{n,0,x}(x) = \frac{x^t}{(n+1)!}x^{n+1} \) for some \( t \in (0, x) \) or \((x, 0)\). In particular then \( |R_{n,0,g}(x)| \leq \frac{3}{(n+1)!}|x|^{n+1} \) for \( x \in (-1, 1) \) and therefore for \( n \geq 1 \)
\[
\frac{e^x - 1}{x} = \sum_{i=1}^{n} \frac{x^{i-1}}{i!} + \frac{1}{x}R_{n,0,g}(x)
\]
\[
= \sum_{k=0}^{n-1} \frac{x^k}{(k+1)!} + \frac{1}{x}R_{n,0,g}(x)
\]
where
\[
\left| \frac{1}{x}R_{n,0,g}(x) \right| \leq \frac{3}{(n+1)!}|x|^n
\]
But for \( n \geq 1 \) the right hand side is continuous at 0 and has a limit of 1 there. Moreover, since \( \frac{1}{x}R_{n,0,g}(x) = O(x^n) \) in the notation from class it follows that
\[
\sum_{k=0}^{n-1} \frac{x^k}{(k+1)!}
\]
is the degree \( n - 1 \) Taylor polynomial of \( f \). Thus \( \frac{d^k}{dx^k} f(x) = \frac{1}{k+1} \). Finally
\[
\int_0^1 f = \int_0^1 \sum_{k=0}^{n-1} \frac{x^k}{(k+1)!} + \frac{1}{x}R_{n,0,g}(x) dx
\]
so we just need to find an \( n \) so that
\[
\left| \int_0^1 \frac{1}{x}R_{n,0,g}(x) dx \right| \leq 10^{-4}
\]
but

\[
\left| \int_0^1 \frac{1}{x} R_{n,0,g}(x) \, dx \right| \leq \frac{3}{(n+1)!} \int_0^1 x^n \, dx = \frac{3}{(n+1)!}(n+1)
\]

Choosing \( n = 7 \) suffices to make \( \frac{3}{(n+1)!}(n+1) < 10^{-4} \) so we are done and our approximation is

\[
\int_0^1 \sum_{k=0}^6 \frac{x^k}{(k+1)!} \, dx = \sum_{k=0}^6 \frac{1}{(k+1)(k+1)!}
\]

**Problem.** (20.21) Define \( f(x) = (1+x)^\alpha \). We will show by induction that \( \frac{d^k}{dx^k} (1+x)^\alpha = \prod_{i=0}^{k} (\alpha - i + 1)(1+x)^{\alpha-k} \), which will show that \( P_{n,0}(x) = \sum_{k=0}^{n} \binom{\alpha}{k} x^k \). The base case is clear, so suppose that the result holds. Then

\[
\frac{d^{k+1}}{dx^{k+1}} (1+x)^\alpha = \frac{d}{dx} \prod_{i=0}^{k} (\alpha - i + 1)(1+x)^{\alpha-k}
\]

\[
= \prod_{i=0}^{k} (\alpha - i + 1)(\alpha - k - i + 1)(1+x)^{\alpha-k-1}
\]

\[
= \prod_{i=0}^{k+1} (\alpha - i + 1)(1+x)^{\alpha-(k+1)}
\]

which completes the induction. The Lagrange form of the remainder is

\[
f^{(n+1)}(t) \frac{x^{n+1}}{(n+1)!}
\]

for some \( t \in (0, x) \)

\[
= \frac{x^{n+1}}{(n+1)!} \prod_{i=0}^{(n+1)+1} (\alpha - i + 1)(1+t)^{\alpha-(n+1)}
\]

\[
= \binom{\alpha}{n+1} x^{n+1}(1+t)^{\alpha-(n+1)}
\]

while the Cauchy form is

\[
x \frac{(x-t)^n}{n!} f^{(n+1)}(t)
\]

for some \( t \in (0, x) \). This is

\[
= x \frac{(x-t)^n}{n!} \prod_{i=0}^{(n+1)+1} (\alpha - i + 1)(1+t)^{\alpha-(n+1)}
\]
\[(n + 1) \binom{\alpha}{n+1} x(1+t)^{\alpha-1} \left(\frac{x-t}{1+t}\right)^n \]

and similarly for the case of \(t \in (x, 0)\).