1. Find the leading order uniform asymptotic approximation to the solution of
\[
\varepsilon y'' + (\cosh(x))y' - y = 0, \quad y(0) = y(1) = 1, \quad 0 \leq x \leq 1
\]
in the limit \(\varepsilon \to 0\).

We write \(y\) as an expansion in \(\varepsilon\) as 
\[
y = \sum_{n=0}^{\infty} y_n \varepsilon^n.
\]
Then we plug this into the differential equation and group powers of \(\varepsilon\). This gives the differential equation for the first term \(y_0\) as
\[
cosh(x)y' - y = 0
\]
with the boundary conditions \(y(0) = y(1) = 1\). However, this boundary value problem cannot be solved because \(y' = \frac{y}{\cosh(x)} > 0\) in our domain, so we cannot satisfy the boundary condition. This is a singular perturbation problem.

So, there must be a boundary layer at \(x = 0\). Then the solution away from the boundary layer satisfies
\[
cosh(x)y'_\text{out} - y_\text{out} = 0, \quad y(1) = 1
\]
which is solved by
\[
y_\text{out}(x) = e^{\int_x^1 \frac{dy}{\cosh(t)}}
\]
In the inner region, we neglect the term \(-y\) because \(y', y'' >> y\). Furthermore, we have \(0 < x < \delta\) where \(\delta << 1\) is the width of the boundary layer, so \(\cosh(x) \approx 1\). We then can solve
\[
\varepsilon y''_\text{in} + y'_\text{in} = 0, \quad y_\text{in}(0) = 1
\]
This is solved by \(y_\text{in} = 1 + C(e^{-x/\varepsilon} - 1)\) where we can determine \(C\) by matching the inner and outer solutions. The fact that \(y_\text{in}\) varies the most when \(x \sim \varepsilon\), we know that \(\delta \sim \varepsilon\). Then for \(x\) in some region around the edge of the boundary layer, we need to match our solutions. Looking at \(x = O(\varepsilon^{1/2})\), we have
\[
y_\text{in} = 1 - C
\]
and
\[ y_{\text{out}} = e^{\int_0^1 \frac{dt}{\cosh(t)}} \]
so we choose \( C = 1 - e^{\int_0^1 \frac{dt}{\cosh(t)}} \). Putting our solutions together and subtracting the extra from the matching region, the first order uniform approximation is
\[ y_{\text{uni}f} = e^{\int_2^1 \frac{dt}{\cosh(t)}} + 1 + (1 - e^{\int_0^1 \frac{dt}{\cosh(t)}})(e^{-\varepsilon} - 1) - (1 - e^{\int_0^1 \frac{dt}{\cosh(t)}}) \]

2. Consider the equation
\[ \varepsilon^2 y'' = Q(x)y \]
In a WKB expansion we use the form
\[ y(x) \sim \exp \left[ \frac{1}{\delta} \sum_{n=0}^{\infty} \delta^n S_n(x) \right] \]
where \( \delta \) depends on \( \varepsilon \) appropriately. The physical optics approximation consists of keeping only the \( S_0(x) \) and \( S_1(x) \) terms in the limit \( \varepsilon \to 0^+ \). Find the most general form of \( Q(x) \) for which physical optics is actually exact.

Plugging the WKB expansion into the equation gives
\[ \varepsilon^2 y \left[ \frac{1}{\delta} \sum_{n=0}^{\infty} \delta^n S_n'' + \left( \frac{1}{\delta} \sum_{n=0}^{\infty} \delta^n S_n' \right)^2 \right] = Q(x)y \]
which is
\[ \frac{\varepsilon^2 (S_0')^2}{\delta^2} + \frac{\varepsilon^2}{\delta} (S_0'' + 2S_0'S_1') + \varepsilon^2 (S_1'' + (S_1')^2) + 2S_0'S_2' + \ldots = Q(x) \]
Now we assume that \( Q(x) \gg \varepsilon \), then by dominant balance \( \varepsilon \sim \delta \) and
\[ (S_0')^2 = Q(x) \]
Then we also have that
\[ S_0'' + 2S_0'S_1' = 0 \]
and
\[ S_{n-1}' + 2S_0'S_n' + \sum_{j=1}^{n-1} S_j'S_{n-j} = 0 \quad n \geq 2 \]
Now if \( S_2' = 0 \) and \( S_0' \neq 0 \), we have that
\[ 2S_0'S_3' + S_2'' + 2S_2'S_1 = 0 \Rightarrow S_3' = 0 \]
and it is easy to see the induction step as well. Thus, if \( S_2' = 0 \), then \( S_n' = 0 \) for \( n \geq 2 \). We can see that \( S_0'(x) = \sqrt{Q(x)} \) and
\[ S_1' = \frac{-S_0'''}{2S_0'} = \frac{-Q'(x)}{4Q(x)} \]
and then finally
\[ S'_2 = -\frac{1}{2} \left( \frac{(S'_1)^2 + S''_1}{S'_0} \right) = -\frac{1}{32} \left( \frac{5 \left( \frac{Q'(x)}{Q(x)} \right)^2 - 4 \left( \frac{Q''(x)}{Q(x)} \right)}{Q^{1/2}(x)} \right) \]

Then the physical optics approximation is exact if
\[ 5 \left( \frac{Q'(x)}{Q(x)} \right)^2 - 4 \left( \frac{Q''(x)}{Q(x)} \right) = 0 \]
or
\[ 5(Q'(x))^2 - 4Q(x)Q''(x) = 0 \]

3. Consider the system of ODEs:
\[ \dot{x} = yz \quad \dot{y} = -2xz \quad \dot{z} = xy \]

(a) Find a quadratic invariant of these equations.
I assume this means find an energy-like function? Then is
\[ H(x, y, z) = x^2 + y^2 + z^2 \]
we have that
\[ \dot{H} = 2x\dot{x} + 2y\dot{y} + 2z\dot{z} = 2xyz - 4xyz + 2xyz = 0 \]

(b) Assuming that the invariant in (a) is equal to a fixed positive constant of your choice, locate, classify, and examine the stability of the critical points.
The critical points of the system are all the points on any of the three axis (ie any point \((x, 0, 0), (0, y, 0)\) or \((0, 0, z)\), including the origin. The invariant tells us that solutions curves stay on a sphere of radius \(r\) where \(H(x_0, y_0, z_0) = r^2\).
Linearization gives the matrix
\[ A = \begin{pmatrix} 0 & z & y \\ -2z & 0 & -2x \\ y & x & 0 \end{pmatrix} \]

Fixed points on the \(x\) and \(z\) axis have eigenvalues with 0 real part and non zero imaginary part perpendicular to the axis (eigenvalues are \(0, \pm r\sqrt{2}i\). This means that there is rotation around these points. The fixed points on the \(y\) axis are saddle points with local stable subspace in the direction of \((-1, 0, 1)\) for \(y = r\) and \((1, 0, 1)\) for \(y = -r\), local unstable subspace in the directions \((1, 0, 1)\) for \(y = r\) and \((-1, 0, 1)\) for \(y = -r\), and central subspace along the axis. The dynamics are reduced to the two dimensional surface of the sphere \(x^2 + y^2 + z^2 = r^2\).

4. Consider the viscous Burgers equation \(u_t + uu_x = \varepsilon u_{xx}\).
(a) Find the traveling wave solution of the form \( u(x, t) = w(x - st) \), where \( s \) is the constant propagation speed and where \( u(x, t) \to u_l \) as \( x \to -\infty \) and \( u(x, t) \to u_r \) as \( x \to +\infty \).

Using the wave solution gives the ODE
\[
\varepsilon w'' + sw' - ww' = 0
\]
we can integrate this equation to see that
\[
\varepsilon w' + sw - \frac{1}{2}w^2 = A
\]
for some constant \( A \). Then
\[
\varepsilon w' = \frac{1}{2}w^2 - sw + A
\]
we can write this as
\[
\frac{w'}{(w - \bar{w}_1)(w - \bar{w}_2)} = \frac{1}{2\varepsilon}
\]
Where \( \bar{w}_2 = s + \sqrt{s^2 - 2A} \) and \( \bar{w}_2 = s - \sqrt{s^2 - 2A} \). Then we can integrate. We need to use partial fractions. Let \( \bar{w} = \bar{w}_1 - \bar{w}_2 \), then
\[
\int \frac{dw}{w - \bar{w}_1}(w - \bar{w}_2) = \int \left( \frac{1}{\bar{w}(w - \bar{w}_1)} - \frac{1}{\bar{w}(w - \bar{w}_2)} \right) dw = \frac{1}{\bar{w}} \log \left( \frac{w - \bar{w}_1}{w - \bar{w}_2} \right)
\]
and so finally (skipping a bunch of algebra) we get
\[
w = \frac{\bar{w}_1 - \bar{w}_2 Be^{(s-st)\bar{w}/2\varepsilon}}{1 - Be^{(s-st)\bar{w}/2\varepsilon}}
\]
where \( B \) is some constant.

Then we have that \( u_l = \bar{w}_1 \) and \( u_r = \bar{w}_2 \).

(b) What is the relationship between \( u_l \), \( u_r \), and \( s \)?
Notice that
\[
\frac{u_l^2 - u_r^2}{2(u_l - u_r)} = \frac{4s\sqrt{s^2 - 2A}}{2(2\sqrt{s^2 - 2A})} = s
\]
The flux in this equation is \( f(u) = \frac{u^2}{2} \), so this is the RH jump condition.

5. Derive the solution to the heat equation on the semi-infinite domain
\[
u_t - u_{xx} = 0, \quad 0 \leq x < \infty, \quad 0 \leq t < \infty
\]
with zero initial condition, \( u(x, 0) = 0 \), and a prescribed boundary condition at \( x = 0 \):
\[
u(0, t) = g(t), \quad t > 0
\]
also assume that \( u(x, t) \to 0 \) as \( x \to \infty \).
We have a non homogeneous, unforced, Dirichlet problem. We first convert this to a homogeneous, forced problem, then we use Duhamel’s principle to solve. Define

\[ v(x, t) = u(x, t) - g(t) \]

then the equation becomes

\[ v_t - v_{xx} = -g'(t) \]

with initial condition \( v(x, 0) = -g(0) = 0 \) and boundary condition \( v(0, t) = 0 \). Now we define a family of functions \( v(x, t; s) \) which solve

\[ v_t(x, t; s) - v_{xx}(x, t; s) = 0, \quad v(x, s; s) = -g'(s), \quad v(0, t; s) = 0 \]

we solve this with method of images by using an odd extension of \( v(x, s; s) \):

\[ \tilde{g} = \begin{cases} 
-g'(s) & x > 0 \\
g'(s) & x < 0 
\end{cases} \]

The fundamental solution of the heat equation in 1 dimension is

\[ \Phi(x, t) = \frac{1}{(4\pi t)^{1/2}} e^{-x^2/4t} \]

so the solution to this is

\[ v(x, t; s) = \int_{-\infty}^{0} g'(s) \Phi(x - y, t - s)dy - \int_{0}^{\infty} g'(s) \Phi(x - y, t - s)dy \]

\[ = \int_{0}^{\infty} g'(s) \left[ \Phi(x + y, t - s) - \Phi(x - y, t - s) \right] dy \]

Now we can use Duhamel’s principle to find the solution

\[ v(x, t) = \int_{0}^{t} v(x, t; s)ds \]

and so

\[ u(x, t) = \int_{0}^{t} v(x, t; s)ds + g(t) \]

\[ = -\int_{0}^{t} \int_{0}^{\infty} g'(s) \left[ \Phi(x - y, t - s) + \Phi(x + y, t - s) \right] dyds + g(t) \]

6. A PDE is well posed if (I) a solution exists, (II) the solution is unique, and (III) the solution depends continuously on initial conditions. Consider the backward heat equation \( u_t = -u_{xx} \) for \(-\infty < x < \infty\). Use the following steps to show that condition (III) is not satisfied and hence the backward heat equation is ill-posed.
(a) Find the unique solution $u(x,t)$ with the initial condition $u(x,0) = 0$. This solution is $u(x,t) = 0$.

(b) Find a sequence $\{u_n(x,t)\}_{n=1}^\infty$ of solutions such that

i. The initial conditions $u_n(x,0)$ converge to $0$ in the $L^2$ norm as $n \to \infty$ (and hence converge to the initial condition from part (a)).

ii. The solutions $u_n(x,T)$ at time $T$ have $L^2$ norms that tend to $\infty$ as $n \to \infty$ (and hence diverge away from the solution from part(a)).

We know that if $u_n(x,0) = a_n \sin(\lambda_n x)$

the solution (by separation of variables) is $u_n(x,t) = a_ne^{\lambda^2_n t} \sin(\lambda x)$

we can construct our desired sequence through an appropriate choice of $\lambda_n$ and $a_n$. Choose $a_n = e^{-\lambda^2_n T/2}$

then

$u_n(x,0) = e^{-\lambda^2_n T/2} \sin(\lambda_n x) \to 0$ in $L^2$, and

$u_n(x,T) = e^{\lambda^2_n T/2} \sin(\lambda_n x) \to \infty$ in $L^2$ for some increasing sequence of $\lambda_n$ such as $\lambda_n = 2\pi n$. Then we have our sequence of solutions.

January 2012

1. This problem is a simply model for diffraction of light passing through infinitesimally small slits separated by a distance $2a$.

Solve the differential equation

$u_t = \frac{i\lambda}{4\pi} u_{xx}$

with initial source $u(x,0) = f(x) = \delta(x-a) + \delta(x+a), \ a > 0$.

Show that the solution $u(x,t)$ oscillates wildly, but that the intensity $|u(x,t)|^2$ is well behaved.

This is the heat equation. We can use a Fourier transform to solve:

$\hat{u} = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} u e^{-ikx} \, dx$

so we have (this should be familiar)

$\hat{u}_t = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} u_t e^{-ikx} \, dx = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \frac{i\lambda}{4\pi} u_{xx} e^{-ikx} \, dx$
then we integrate by parts twice to see that
\[ \hat{u}_t = -k^2 \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} u e^{-ikx} dx = -ik^2 \frac{\lambda}{4\pi} \hat{u} \]

this is an easy ODE, its solution is
\[ \hat{u} = \hat{u}(k,0) e^{-k^2 \frac{\lambda}{4\pi} t} \]

where \( \hat{u}(k,0) = \hat{u}(x,0) = \sqrt{\frac{2}{\pi}} \cos(ka) \). The solution can be written as the integral
\[ u(x,t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \cos(ak) e^{ikx + \frac{-ik^2 \lambda}{4\pi} t} dk \]
or using convolution \( u(x,0) * \mathcal{F}^{-1} \left( e^{-k^2 \frac{\lambda}{4\pi} t} \right) \) which is
\[ u(x,t) = \mathcal{F}^{-1} \left( e^{-k^2 \frac{\lambda}{4\pi} t} \right) (x-a) + \mathcal{F}^{-1} \left( e^{-k^2 \frac{\lambda}{4\pi} t} \right) (x+a) \]

Notice that this means that \( \|u\|_{L^2} \) behaves nicely by Plancherel’s theorem because \( \|e^{-k^2 \frac{\lambda}{4\pi} t}\|_{L^2} \) is finite and shrinks with time.

2. (a) Solve:
\[ \frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2}, \quad x \geq 0, \quad t \geq 0 \]
\[ u(x,0) = f(x), \quad \frac{\partial u}{\partial t}(x,0) = g(x), \quad \frac{\partial u}{\partial x}(0,t) = h(t) \]

To solve this, we need the transform:
\[ u = w + xh(t) \]

which gives the forced equation
\[ w_{tt} - c^2 w_{xx} = -xh''(t) \]
\[ w(x,0) = f(x) - xh(0), \quad w_t(x,0) = g(x) - xh'(0), \quad w_x(0,t) = 0 \]

Claim: Define \( u(x,t;s) \) as a solution to
\[ u_{tt}(x,t;s) - c^2 u_{xx}(x,t;s) = 0 \]
\[ u(x,s;s) = 0, \quad u_t(x,s;s) = h(x,s), \quad u_x(0,t;s) = 0 \]

then
\[ u_{tt}(x,t) - c^2 u_{xx}(x,t) = h(x,t) \]
\[ u(x,0) = f(x), \quad u_t(x,0) = g(x), \quad u_x(0,t) = 0 \]
is solved by
\[
    u(x, t) = \begin{cases} 
    \int_0^t u(x, t; s)ds + \frac{1}{2} (f(x + ct) + f(x - ct)) + \frac{1}{2c} \int_{x-ct}^{x+ct} g(\xi) d\xi & x > ct \\
    \int_0^t u(x, t; s)ds + \frac{1}{2} (f(x + ct) + f(ct - x)) + \frac{1}{2c} \int_{ct-x}^{ct+x} g(\xi) d\xi & x < ct 
    \end{cases}
\]
This satisfies the initial condition \( u(x, 0) = f(x) \), and it satisfies \( u_t(x, 0) = g(x) \) because at \( t = 0, x \geq ct \), so
\[
    \frac{\partial}{\partial t} \int_0^t u(x, t; s)ds = \int_0^t u_t(x, t; s)ds + u(x, t; t) = 0
\]
We also have at \( x = 0 \) that \( x < ct \) (notice we can’t deal with \( f'(0) \))
\[
    u_x(0, t) = \int_0^t u_x(0, t; s)ds + \frac{f'(ct) - f'(ct)}{2} + \frac{1}{2c} (g(ct) - g(ct)) = 0
\]
Finally, a calculation shows that
\[
    u_{tt} = h(x, t) + \int_0^t u_{tt}(x, t; s)ds + \frac{c^2}{2} (f''(x + ct) + f''(x - ct)) + \frac{c}{2} (g'(x + ct) - g'(x - ct))
\]
and
\[
    u_{xx} = \int_0^t u_{xx}(x, t; s)ds + \frac{1}{2} (f''(x + ct) + f''(x - ct)) + \frac{1}{2c} (g'(x + ct) - g(x - ct))
\]
so
\[
    u_{tt} - c^2 u_{xx} = h(x, t)
\]
Now that we know that, we can attack our problem. We need to first solve
\[
    w_{tt}^* - c^2 w_{xx}^* = 0, \quad w^*(x, s; s) = 0, \quad w_t^*(x, s; s) = -x h''(s), \quad w^*_2(0, t; s) = 0
\]
This is solved by
\[
    w^*(x, t; s) = \begin{cases} 
    -\frac{1}{2c} \int_{x-c(t-s)}^{x+c(t-s)} \xi h''(s) d\xi = -x(t-s)h''(s) & x > c(t-s) \\
    -\frac{1}{2c} \int_{x-c(t-s)}^{x+c(t-s)} |\xi| h''(s) d\xi = -\frac{1}{2c} (x^2 + c^2(t-s)^2) h''(s) & x < c(t-s) 
    \end{cases}
\]
then we have that
\[
    w(x, t) = \begin{cases} 
    \int_0^t w^*(x, t; s)ds + \frac{f(x + ct) - 2x h(0) + f(x + ct)}{2} + \frac{1}{2c} \int_{x-ct}^{x+ct} (g(\xi) - \xi h'(0)) d\xi & x > ct \\
    \int_0^t w^*(x, t; s)ds + \frac{f(x + ct) - 2x h(0) + f(x + ct)}{2} + \frac{1}{2c} \int_{ct-x}^{ct+x} (g(\xi) - \xi h'(0)) d\xi & x < ct 
    \end{cases}
\]
and finally
\[
    u(x, t) = w(x, t) + x h(t)
\]
(b) For the special case $h(t) = 0$, explain how you could use a symmetry argument to help construct the solution.

In that case, we notice that if extend $f(x)$ and $g(x)$ in an even manner, then this is preserved in time. This means that the solution to the problem on the whole line with an even extension of these initial conditions satisfies this half line problem.

3. Find a lowest order uniform approximation to the boundary value problem

$$\varepsilon y'' + \sin(x)y' + \sin(2x)y = 0, \quad y(0) = \pi, \quad y(\pi) = 0$$

To begin, we will need to solve the zero-order ODE:

$$\sin(x)y' + \sin(2x)y = 0$$

It is likely that we will have a singular problem. We will know we have this if the zero-order solution cannot satisfy these boundary conditions. Indeed, the zero-order solution is

$$y(x) = Ae^{-2\sin(x)}$$

which cannot satisfy both boundary conditions simultaneously. This can be solved with a boundary layer. The zero order solution is the solution outside of the boundary layer. If we take the boundary layer to be on the left, we get the outer solution $y_{\text{out}} = 0$. Then we can solve in the boundary layer by setting $x = Z\delta$. Under this transformation, we have

$$\varepsilon \frac{\delta^2}{\delta^2} y'' + \frac{\sin(\delta Z)}{\delta} y' + \sin(2\delta Z)y = 0$$

which to first order is

$$\varepsilon \frac{\delta^2}{\delta^2} y'' + Zy' = 0$$

This tells us that the boundary layer has thickness $\delta = \varepsilon^{1/2}$, and we need to solve

$$y'' + Zy' = 0$$

This is solved by

$$y = \text{Aerf}(Z) + B$$

Using the boundary condition $y(0) = \pi$ this gives us

$$y(x) = \text{Aerf} \left( \frac{x}{\varepsilon^{1/2}} \right) + \pi$$

We then need to match this as we approach the edge of the boundary layer. This means that for $x > \varepsilon^{1/2}$ we need to go to $y = 0$. We can solve for $A$:

$$A = -\pi$$
Then our uniform approximation is

$$y_{unif}(x) = -\pi \text{erf} \left( \frac{x}{\varepsilon^{1/2}} \right) + \pi$$

If we put the boundary layer at $y = \pi$, we have the outer solution satisfying $y(0) = \pi$, so

$$y_{out} = \pi e^{-2\sin(x)}$$

Then to solve in the boundary layer we use the transformation $x = \pi - Z\delta$. We then have

$$\varepsilon \frac{\delta^2 y''}{\delta^2} - \frac{\sin(\pi - Z\delta)}{\delta} y' + \sin(2(\pi - Z\delta)) = 0$$

again, $\delta = \varepsilon^{1/2}$ and near $x = \pi$ we have

$$y'' - Zy' = 0$$

Again this is solved by

$$y = A \int_{0}^{Z} e^{-t^2} dt + B$$

and using the boundary condition $y(\pi) = 0$, we have

$$y_{in} = A \int_{0}^{\pi^{1/2}} e^{-t^2} dt$$

This cannot be matched.

4. Find the value(s) of the constant $c$ for which the 2D boundary value problem

$$\nabla^2 u = c, \quad x^2 + y^2 < 1$$

with

$$\frac{\partial u}{\partial r} = 2 \quad \text{on} \quad r = 1$$

has a solution. Then find the general regular solution.

Since we are working on a disk, it will be helpful to use polar coordinates. The Laplace operator in polar coordinates is

$$\nabla^2 u = u_{rr} + \frac{1}{r} u_r + \frac{1}{r^2} u_{\theta\theta}$$

so we have

$$u_{rr} + \frac{1}{r} u_r + \frac{1}{r^2} u_{\theta\theta} = c$$

Next we can use a simple transform to shift our constant forcing over to our boundary condition. That is

$$\tilde{u} = u - bn^2$$
then we have
\[ \ddot{u}_{rr} + \frac{1}{r} \dot{u}_r + \frac{1}{r^2} \ddot{u}_{\theta\theta} + 4b = c \]
Then if we choose \( b = \frac{c}{2} \), we have
\[ \nabla^2 u = 0 \]
with the boundary condition
\[ \frac{\partial \ddot{u}}{\partial r}(1, \theta) = 2 - 2b = 2 - \frac{c}{2} \]
We can solve by separation of variables, \( \ddot{u} = T(\theta)R(r) \), which gives the ODEs
\[ T''(\theta) = -\lambda^2 T(\theta) \]
so we have that
\[ T(\theta) = A_n \cos(\lambda_n \theta) + B_n \sin(\lambda_n \theta) \]
We know that our solution must be periodic in \( \theta \), so we have that \( \lambda_n = n \) for integers \( n \).
Next,
\[ r^2 R''(r) + r R'(r) = \lambda^2 R \]
which gives the solution
\[ R(r) = r^{\lambda_n} + r^{-\lambda_n} \]
for \( \lambda_n \neq 0 \) and
\[ R(r) = C + \ln(r) \]
for \( \lambda_n = 0 \). Assuming boundedness as \( r \to 0 \), we then have
\[ R(r) = r^n \]
Putting these together we see that
\[ \ddot{u} = \sum_{n=0}^{\infty} r^n (A_n \cos(n\theta) + B_n \sin(n\theta)) \]
and we have that
\[ \frac{\partial \ddot{u}}{\partial r}(1, \theta) = \sum_{n=1}^{\infty} n r^{n-1} (A_n \cos(n\theta) + B_n \sin(n\theta)) = 2 - \frac{c}{2} \]
we can then integrate and we see that
\[ nA_n \pi = \left(2 - \frac{c}{2}\right) \int_0^{2\pi} \cos(n\theta) d\theta = 0 \]
and
\[ nB_n \pi = \left(2 - \frac{c}{2}\right) \int_0^{2\pi} \sin(n\theta) d\theta = 0 \]
so for \( n > 0 \), we have

\[ A_n = B_n = 0 \]

The solution is then \( \tilde{u} = A_0 \), so

\[ \frac{\partial \tilde{u}}{\partial r}(1, \theta) = 0 \]

so \( 2 - \frac{c}{2} = 0 \), so we know that \( c = 4 \).

We also can get our solution now. We know

\[ u(r, \theta) = \tilde{u} + br^2 = A_0 + r^2 \]

so we have that

\[ u(x, y) = A_0 + x^2 + y^2 \]

5. Suppose that

\[ \frac{\partial u}{\partial t} + \frac{\partial u}{\partial x} = \delta \left( x - \frac{1}{2} t \right) \], \quad t > 0, \quad u(x, 0) = 0 \]

Reformulate the problem by changing variables from \((x, t)\) to \((\xi, t)\) where \( \xi = x - \frac{1}{2} t \).

Then solve the problem using the method of characteristics.

Making this change of variables we see that

\[ \frac{\partial u}{\partial t} = \frac{\partial u}{\partial \xi} + u_t = u_t - tu_\xi \]

\[ \frac{\partial u}{\partial x} = \frac{\partial u}{\partial \xi} \frac{\partial \xi}{\partial x} = u_\xi \]

and so we have the equation

\[ u_t - (1 - t)u_\xi = \delta(\xi), \quad u(\xi, 0) = 0 \]

We need to use Duhamel’s principle to solve this forced equation. We define \( u(\xi, t; s) \) as a solution to

\[ u_t - (1 - t)u_\xi = 0, \quad u(\xi, s; s) = \delta(\xi) \]

We can solve this with the method of characteristics. We have that if

\[ u(\xi, t; s) = u(\xi(t), t; s) \]

then

\[ \frac{d}{dt}u(\xi(t), t; s) = u_t + \xi'(t)u_\xi = 0 \]

if \( \xi'(t) = 1 - t \). Then \( u(\xi(t), t) \) is constant in time. We need only to find the characteristics \( \xi(t) \) such that \( \xi'(t) = 1 - t \). These are

\[ \xi(t) = t - \frac{t^2}{2} - s + \frac{s^2}{2} + \xi(s) \]

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so we have that
\[ u(\xi(t), t; s) = u(\xi(s), s; s) = \delta(\xi(s)) \]
and so
\[ u(\xi, t; s) = \delta \left( \xi + s - t + \frac{t^2}{2} - \frac{s^2}{2} \right) \]
and so the solution is
\[ \int_0^t \delta \left( x - t + s - \frac{s^2}{2} \right) ds \]

6. A piston at \( x = X(t) \) oscillates with small amplitude about \( x = 0 \) in a tube under the action of a pressure \(-p\). The model is
\[ \frac{d^2X}{dt^2} + \Omega^2 X = \nu p \]
where \( \Omega \) and \( \nu \) are constants and \( p \) is given by the acoustic model:
\[ p = \left. \frac{\partial \phi}{\partial t} \right|_{x=0}, \quad \frac{dX}{dt} = \left. \frac{\partial \phi}{\partial x} \right|_{x=0}, \quad \frac{\partial^2 \phi}{\partial t^2} - a_0^2 \frac{\partial^2 \phi}{\partial x^2}, \quad x > 0 \]
where \( a_0 \) is the speed of sound and \( \phi \) is the velocity potential. Find the dispersion relation for outgoing waves as \( x \to \infty \). How do you interpret a complex frequency of wave speed?

The dispersion relation from the wave equation is just
\[ \omega(k) = \pm ak \]
We see this by plugging a plane wave solution \(-e^{i(kx-\omega t)}\) - into equation.

1 **August 2011**

1. The radius \( r(t) \) of a cloud droplet grows in time approximately as \( \frac{dr}{dt} = \frac{s}{r} \). Here \( s > 0 \) is taken to be a positive constant that represents the “supersaturation”, i.e. the extent to which the ambient air is above the saturation point (100% humidity). A population of cloud droplets can be represented by a density function \( f(r, t) \), which gives the number of droplets with radius between \( r \) and \( r + dr \) at time \( t \). The density function \( f(r, t) \) then evolves according to
\[ \frac{\partial f}{\partial t} + \frac{\partial}{\partial r} \left( s \frac{r}{r^2} f \right) = 0, \quad f(r, t)|_{t=0} = f_0(r) \]
Find a formula for \( f(r, t) \) in terms of the initial profile \( f_0(r) \). Assume that \( f_0(r) \) vanishes rapidly as \( r \to 0 \) and \( r \to \infty \).

First I will claim that any function of the form
\[ f(r, t) = rg \left( \frac{r^2}{2s} - t \right) \]
is a solution to the ODE. Differentiating, we have
\[
\frac{\partial f}{\partial t} = -rg'
\]
and
\[
\frac{\partial}{\partial r} \left( \frac{s}{r} f \right) = \frac{\partial}{\partial r} \left( sg \left( \frac{r^2}{2s} - t \right) \right) = rg'
\]
so
\[
\frac{\partial f}{\partial t} + \frac{\partial}{\partial r} \left( \frac{s}{r} f \right) = rg' - rg' = 0
\]
Then we need
\[
f_0(r) = rg \left( \frac{r^2}{2s} \right)
\]
so we see that
\[
g(x) = \frac{1}{\sqrt{2sx}} f_0 \left( \sqrt{2xs} \right)
\]
and so
\[
f(r, t) = \frac{r}{\sqrt{r^2 - 2st}} f_0 \left( \sqrt{r^2 - 2st} \right)
\]
but this solution is only valid for \( r^2 > 2st \). For \( r^2 < 2st \), we notice we can just flip the sign. Then the solution is
\[
f(r, t) = \frac{r}{\sqrt{|r^2 - 2st|}} f_0 \left( \sqrt{|r^2 - 2st|} \right)
\]

2. Determine the exact entropy solution to Burgers’ equation \( u_t + \left( \frac{1}{2} u^2 \right)_x = 0 \) for all \( t > 0 \) with the initial data
\[
u(x, 0) = \begin{cases} 
1 & x < -1 \\
0 & -1 < x < 1 \\
-1 & x > 1
\end{cases}
\]
One entropy solution (there are other ways to define this...) is to use the RH jump conditions. We have two shocks here, the one on the left travels at the speed
\[
s_1 = \frac{f(1) - f(0)}{1 - 0} = \frac{1}{2}
\]
and the one on the right moves at the speed
\[
s_2 = \frac{f(0) - f(-1)}{0 - (-1)} = -\frac{1}{2}
\]
so they will meet at time \( t = \frac{1}{2} \) at \( x = 0 \). After time \( t = \frac{1}{2} \), there will be one shock moving at speed 0.
3. Calculate and match the outer and inner expansions to $O(1)$ for

$$
\varepsilon u'' + \frac{u'}{x^{1/2}} - u = 0, \quad 0 \leq x \leq 1, \quad 0 < \varepsilon << 1
$$

$$
u(0; \varepsilon) = 0, \quad u(1; \varepsilon) = e^{\frac{3}{4}}
$$

Find the overlap region explicitly.

This is a singular problem. The boundary layer should be at $x = 0$, because $\frac{1}{x^{1/2}} > 0$ on $0 \leq x \leq 1$. We will see that this is indeed the case when our matching works out.

The outer solution solves

$$
u' = x^{1/2} u, \quad u(1) = e^{\frac{3}{4}}
$$

this is solved by

$$
u_{out} = e^{\frac{3}{4} x^{3/2}}
$$

To solve in the boundary layer, it is helpful to rescale as $x = Z\delta$, where $Z = O(1)$. Then we have the problem

$$
\varepsilon \frac{\delta^2}{\delta} u''(Z) + \frac{1}{\delta} \frac{u'(Z)}{Z^{1/2}\delta^{1/2}} - u(Z) = 0
$$

Dominant balance (and the assumption that $u''$, $u' >> u$ in the boundary layer) then tell us that

$$
\delta = \varepsilon^2
$$

and for our first order approximation we need to solve the ODE

$$
u'' = -\frac{u'}{Z^{1/2}}, \quad u(0) = 0
$$

One integration gives

$$
u' = Ae^{-2Z^{1/2}}
$$

so

$$
u = A \int_0^Z e^{-2t^{1/2}} \, dt
$$

Making the substitution $w = -2t^{1/2}$, we have the integral

$$
A \int_0^t e^w \, dw = A(w - 1)e^{w|t=Z} = A(2Z^{1/2} + 1) e^{-2Z^{1/2}} + C
$$

And the boundary condition tells us that $C = -A$. Finally we have that

$$
u_{in} = A \left[ \left( \frac{2x^{1/2}}{\varepsilon} + 1 \right) e^{-2x^{1/2}} - 1 \right]
$$

Now we can match the inner and outer solutions. As we leave the boundary layer, we have that

$$
u_{in} \to -A
$$
and as we enter the boundary layer,
\[ u_{out} \rightarrow 1 \]
so we have a matching if \( A = -1 \). Then
\[ u_{in} = 1 - \left( \frac{2x^{1/2}}{\varepsilon} + 1 \right) e^{-2x^{1/2}} \]
and the uniform approximation is (subtracting off the overlap)
\[ u_{unif} = u_{out} + u_{in} - 1 = e^{\frac{2x^{3/2}}{\varepsilon}} - \left( \frac{2x^{1/2}}{\varepsilon} + 1 \right) e^{-2x^{1/2}} \]
The overlap region is about \( \varepsilon^2 < x < \varepsilon^{1/2} \)

4. Solve
\[ \frac{\partial u}{\partial t} = \nu \frac{\partial^2 u}{\partial x^2} + Q(x, t) \]
\[ u(x, 0) = f(x), \quad u(0, t) = A(t), \quad u(L, t) = B(t) \]
To attack this problem, we can start with a homogenizing transform to get homogeneous boundary conditions. This is
\[ w(x, t) = u(x, t) - A(t) - \frac{x}{L} (B(t) - A(t)) \]
Then we have that
\[ w(0, t) = u(0, t) - A(t) = 0 \]
and
\[ w(L, t) = u(L, t) - B(t) = 0 \]
However, this does add complication to the problem by adding to our forcing term (although because there was already a forcing term, this isn’t really a big deal). We have that \( w_{xx} = u_{xx} \), but
\[ u_t = A'(t) + \frac{x}{L} (B'(t) - A'(t)) + w_t \]
so
\[ w_t = \nu w_{xx} + Q(x, t) - A'(t) - \frac{x}{L} (B'(t) - A'(t)) \]
and we have the initial condition
\[ w(x, 0) = f(x) - A(0) - \frac{x}{L} (B(0) - A(0)) \]
To solve this, we first need the solution to the unforced problem. The solution is
\[ w_{unforced} = \sum_n \left( A_n \cos \left( \frac{\lambda_n}{\sqrt{\nu}} x \right) + B_n \sin \left( \frac{\lambda_n}{\sqrt{\nu}} x \right) \right) e^{-\lambda_n^2 t} \]
The Dirichlet boundary conditions tell us that \( A_n = 0 \) for all \( n \) and that \( \lambda_n = \frac{n\pi}{L} \). The solution is then

\[
w_{\text{unforced}} = \sum_n B_n e^{-\frac{n^2 \pi^2 v t}{L^2}} \sin \left( \frac{n\pi}{L} x \right)
\]

we can find our coefficients \( B_n \) with the initial condition, taking advantage of the orthogonality of the set \( \{ \sin \left( \frac{n\pi}{L} x \right) \} \). We thus have

\[
B_n = 2 \frac{L}{L} \int_0^L f(x) \sin \left( \frac{n\pi}{L} x \right) \, dx + (2(-1)^{n+1} - 1) \frac{2}{n\pi} A(0) + (-1)^{n+1} \frac{2}{n\pi} B(0)
\]

Now, we can solve the forced problem using Duhamel’s principle. Define \( w(x, t; s) \) as a solution to

\[
w_t = \nu w_{xx}
\]

with homogeneous boundary conditions and the initial condition

\[
w(x, s; s) = Q(x, s) = A'(s) - \frac{x}{L} (B'(s) - A'(s))
\]

The solution (just like before) is

\[
w(x, t; s) = \sum_n C_n e^{-\frac{n^2 \pi^2 v t}{L^2}} \sin \left( \frac{n\pi}{L} x \right)
\]

where

\[
C_n = e^{\frac{-n^2 \pi^2 v s}{L^2}} \left[ 2 \frac{L}{L} \int_0^1 Q(x, s) \sin \left( \frac{n\pi}{L} x \right) \, dx + (2(-1)^{n+1} - 1) \frac{2}{n\pi} A'(s) + (-1)^{n+1} \frac{2}{n\pi} B'(s) \right]
\]

so we have

\[
w(x, t; s) = \sum_n A_n e^{\frac{-n^2 \pi^2 v (s-t)}{L^2}} \sin \left( \frac{n\pi}{L} x \right)
\]

where

\[
A_n = 2 \frac{L}{L} \int_0^1 Q(x, s) \sin \left( \frac{n\pi}{L} x \right) \, dx + (2(-1)^{n+1} - 1) \frac{2}{n\pi} A'(s) + (-1)^{n+1} \frac{2}{n\pi} B'(s)
\]

and that

\[
w(x, t) = \sum_n \int_0^t A_n e^{\frac{-n^2 \pi^2 v (s-t)}{L^2}} \sin \left( \frac{n\pi}{L} x \right) \, ds + \sum_n B_n e^{-\frac{n^2 \pi^2 v t}{L^2}} \sin \left( \frac{n\pi}{L} x \right)
\]

and finally that

\[
u(x, t) = A(t) + \frac{x}{L} (B(t) - A(t)) + \sum_n \int_0^t A_n e^{\frac{-n^2 \pi^2 v (s-t)}{L^2}} \sin \left( \frac{n\pi}{L} x \right) \, ds + \sum_n B_n e^{-\frac{n^2 \pi^2 v t}{L^2}} \sin \left( \frac{n\pi}{L} x \right)
\]
\[ A_n = \frac{2}{L} \int_0^1 Q(x, s) \sin \left( \frac{n\pi x}{L} \right) dx + \left( 2(\pi - 1) n^{n+1} - 1 \right) \frac{2}{n\pi} A'(s) + (-1)^{n+1} \frac{2}{n\pi} B'(s) \]

\[ B_n = \frac{2}{L} \int_0^L f(x) \sin \left( \frac{n\pi x}{L} \right) dx + \left( 2(\pi - 1) n^{n+1} - 1 \right) \frac{2}{n\pi} A(0) + (-1)^{n+1} \frac{2}{n\pi} B(0) \]

5. Consider the competition model for two species with populations \( N_1 \) and \( N_2 \),

\[ \frac{dN_1}{dt} = r_1 N_1 \left( 1 - \frac{N_1}{K_1} - b_{12} \frac{N_2}{K_1} \right) \]

\[ \frac{dN_2}{dt} = r_2 N_2 \left( 1 - b_{21} \frac{N_1}{K_2} \right) \]

Non-dimensionalize the system and recast the equations in terms of the new variables \( u_1 = \frac{N_1}{K_1} \), \( u_2 = \frac{N_2}{K_2} \) and scaled time \( \tau = r_1 t \). Investigate the stability of this system and sketch the phase plane trajectories. Briefly describe under what conditions the species \( N_2 \) becomes extinct.

Recasting the equations, we have that

\[ \frac{du_1}{d\tau} = \frac{1}{r_1 K_1} \frac{dN_1}{dt} \]

\[ \frac{du_2}{d\tau} = \frac{1}{r_1 K_2} \frac{dN_2}{dt} \]

Plugging in and substituting, we have

\[ \frac{du_1}{d\tau} = u_1 \left( 1 - u_1 - \beta_1 u_2 \right) \]

\[ \frac{du_2}{d\tau} = \rho u_2 \left( 1 - \beta_2 u_1 \right) \]

where \( \beta_1 = \frac{b_{12} K_2}{K_1} \), \( \beta_2 = \frac{b_{21} K_1}{K_2} \), and \( \rho = \frac{r_2}{r_1} \) are non dimensional parameters. There are a lot of cases to go through, but the fixed points the eigenvalues of their linearised system are

\[ (0, 0) \quad \lambda_1 = 1, \lambda_2 = \rho \]

\[ (1, 0) \quad \lambda_1 = -1, \lambda_2 = \rho (1 - \beta_2) \]

\[ \left( \frac{1}{\beta_2}, \frac{\beta_2 - 1}{\beta_2 \beta_1} \right) \quad \lambda_{1,2} = \frac{1}{2\beta_2} \left( -1 \pm \sqrt{1 - 4\rho \beta_2 (\beta_2 - 1)} \right) \]

Where for the last fixed point, \( \Re(\lambda_{1,2}) < 0 \) if \( \beta_2 > 1 \). If \( \beta_2 < 1 \), this fixed point isn’t physically relevant. Species \( N_2 \) can become extinct when \( \beta_2 > 1 \) and the initial amount of \( N_2 \) is below some threshold.

6. Consider the Navier-Stokes equations for the flow of an incompressible Newtonian fluid:

\[ \frac{\partial \mathbf{u}}{\partial t} + \mathbf{u} \cdot \nabla \mathbf{u} = -\frac{1}{\rho} \nabla p + \nu \nabla^2 \mathbf{u} \]

\[ \nabla \cdot \mathbf{u} = 0 \]
Now assume slow flow, i.e., with all accelerations negligible compared to the contribution from viscous stresses. Solve for the flow velocities in the case of slow flow past a sphere. Use appropriate boundary conditions on the surface of the sphere and at infinity.

Nope