Math 234 Exam 1 Review Sheet

LIST OF TOPICS TO KNOW

- Vectors:
  - Adding, subtracting, and scalar multiplication
  - cross product & dot product
    - computation
    - geometry (ANGLES!!!)
  - Equations of lines and planes
  - Distance between objects in space (planes, lines, points)
- Parametric curves
  - parametrizing curves
  - velocity, acceleration (vector, length)
  - unit tangent (computation, geometry)
  - curvature (vector, length)
  - arc length
- Functions of multiple variables
  - Visualization of the function
    - cross sections
    - factoring/completing the square (may be helpful)
  - level sets
    - drawing simple level sets
    - matching level sets with curves
  - partial derivatives (the gradient vector)
  - linear approximation/tangent plane
  - the chain rule in two dimensions

VECTORS

$$\vec{a} = \langle 4, 5, 1 \rangle \qquad \vec{b} = \langle 3, 2, 1 \rangle$$

The dot product $\vec{a} \cdot \vec{b}$ is

$$a \cdot b = 4 * 3 + 5 * 2 + 1 * 1 = 12 + 10 + 1 = 23$$

and can also be computed

$$\vec{a} \cdot \vec{b} = \|\vec{a}\| \|\vec{b}\| \cos(\theta)$$

where θ is the angle between $\vec{a}$ and $\vec{b}$. Here we don't know θ but we can use the dot product to calculate it.

$$\|\vec{a}\| \|\vec{b}\| \cos(\theta) = (42)^{1/2} (15)^{1/2} \cos(\theta) = 23$$

so

$$\theta = \arccos\left(\frac{23}{\sqrt{588}}\right) \approx 0.322 radians$$

The cross product is

$$\vec{a} \times \vec{b} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 4 & 5 & 1 \\ 3 & 2 & 1 \end{vmatrix} = 3\hat{i} - \hat{j} - 7\hat{k}$$

We could also calculate the length of cross product with the formula

$$\|\vec{a} \times \vec{b}\| = \|\vec{a}\| \|\vec{b}\| \sin(\theta) \approx \sqrt{588} \sin(0.322) \approx 7.67$$

So, find the area of the parallelogram with sides $\vec{a}$ and $\vec{b}$. We already have.
What is the equation of the plane containing the points $(1, 1, 1)$, $(3, 3, 3)$ and $(1, 2, 3)$?
We need two vectors that lie in the plane. Two possibilities are $\langle 2, 2, 2 \rangle$ and $\langle 0, 1, 2 \rangle$. Then the normal to the plane is

$$\langle 2, 2, 2 \rangle \times \langle 0, 1, 2 \rangle = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 2 & 2 & 2 \\ 0 & 1 & 2 \end{vmatrix} = 2\hat{i} - 4\hat{j} + 2\hat{k}$$

SANITY CHECK!!!! Is this perpendicular to my vectors??? (Use dot product to show that it is).
My plane is given by

$$2(x - 1) - 4(y - 1) + 2(z - 1) = 0 \Rightarrow 2x - 4y + 2z = 0$$

How far is the point $(2, 0, 1)$ from the plane?
First, let's write a vector from the plane to the point:

$$\vec{v} = \langle 2 - 1, 0 - 1, 1 - 1 \rangle = \langle 1, -1, 0 \rangle$$

We know that

$$\cos(\theta) = \frac{\vec{v} \cdot \vec{n}}{\|\vec{v}\| \|\vec{n}\|}$$

and we can draw some triangles to see that the distance we are looking for, d, is

$$d = |\|\vec{v}\|\cos(\theta)|$$

so

$$d = \left|\frac{\vec{v}\cdot\vec{n}}{\|\vec{v}\|\|\vec{n}\|}\right| = \frac{6}{\sqrt{24}}$$

PARAMETRIC CURVES

Can we define a parametric curve that is an inward, clockwise spiral that ends up at the point $(1,1)$??
OF COURSE WE CAN!!! If we wanted to parametrise a "clockwise" oriented circle, we could use

$$\vec{r}(t) = \langle R\sin t, R\cos t\rangle$$

(this is different from what we normally write down for a circle). Now, we simply need the radius to decrease to 0 as $t \to \infty$ and increase to ∞ as $t \to -\infty$. So we are after $R(t)$ so that $\vec{r}(t) = \langle R(t)\sin t, R(t)\cos t\rangle$ is a spiral. We can use $R(t) = e^{-t}$.
NOTICE - there are other possible choices that would give us a similar looking spiral - these MAY OR MAY NOT be exactly the same spiral - what if I chose $R(t) = \frac{1}{t}$, or $R(t) = e^{-3t}$?????
Our curve is going to be $\vec{r}(t) = \langle e^{-t}\sin t, e^{-t}\cos t\rangle$. We can now ask a lot of questions about the curve.
What is the "velocity" of the curve with respect to t?

$$\vec{r}'(t) = \langle -e^{-t}\sin t + e^{-t}\cos t, -e^{-t}\cos t - e^{-t}\sin t\rangle$$

and the "speed"?

$$\|\vec{r}'(t)\| = \sqrt{e^{-2t}(\cos t - \sin t)^2 + e^{-2t}(\cos t + \sin t)^2} = e^{-t}\sqrt{2}$$

And the tangent direction? (Remember this is the DIRECTION of the rate of change but ignores speed - i.e. it is a unit vector)

$$\vec{T}(t) = \frac{\vec{r}'(t)}{\|\vec{r}'(t)\|} = \frac{1}{e^{-t}\sqrt{2}}\langle -e^{-t}\sin t + e^{-t}\cos t, -e^{-t}\cos t - e^{-t}\sin t\rangle = \frac{-1}{\sqrt{2}}\langle \sin t - \cos t, \cos t + \sin t\rangle$$

What about the curvature, $\vec{\kappa}$?

$$\vec{\kappa} = \frac{\vec{T}'(t)}{\|\vec{r}'(t)\|} = \frac{-1}{e^{-t}2}\langle \cos t + \sin t, \cos t - \sin t\rangle$$

and let's calculated the length of $\vec{\kappa}$

$$\|\vec{\kappa}\| = \left(\frac{1}{2e^{-t}}\right)\sqrt{(\cos t + \sin t)^2 + (\cos t - \sin t)^2} = \frac{\sqrt{2}}{2e^{-t}}$$

GEOMETRIC SANITY CHECK - DOES THIS MAKE SENSE?? YES.
Finally, let's calculate the arc length from where we pass through $(1, 2)$ to when $t = a$. Notice that $\vec{r}(0) = \langle 1, 2\rangle$, so we want

$$\int_0^a \|\vec{r}'(t)\| dt = \int_0^a \sqrt{2}e^{-t} dt = -\sqrt{2}e^{-t}\Big|_0^a = \sqrt{2}(1 - e^{-a})$$

And if we want the length of our spiral from $(1, 2)$ to $(1, 1)$, we see that as $a \to \infty$, we get it. That length is then $\lim_{a\to\infty} \sqrt{2}(1 - e^{-a}) = \sqrt{2}$.

FUNCTIONS OF MULTIPLE VARIABLES

Can we visualize the following function?

$$f(x, y) = 10 - (x^2 + y^2)$$

We might consider inspecting the VERTICAL planes $x = 0$ and $y = 0$ to see that vertical cross sections of this are parabolas, and conclude that this is something like an upside down bowl. What do its level sets look like??

$$a = 9 - (x^2 + y^2) \Rightarrow x^2 + y^2 = 10 - a$$

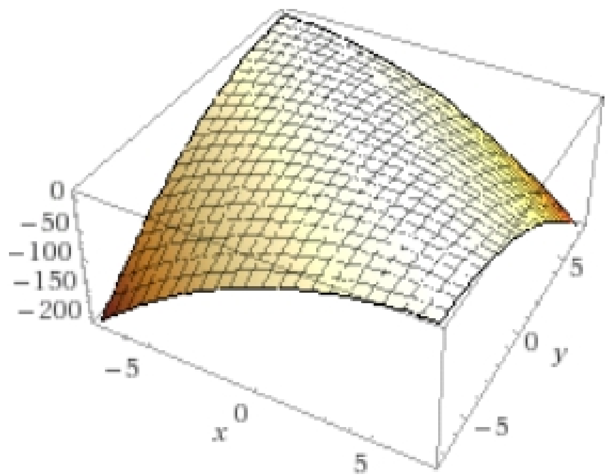
so they are circles of radius $\sqrt{9 - a}$. If $a > 9$, we don't have any level sets - that is not in the range of the function.
GREAT! What about

$$g(x, y) = 9 - (x + y)^2$$

Let's try the same type of tricks. If we "freeze" our variables, we see that vertical cross sections on the x, z and y, z planes are the same as the first example. HOWEVER!!!!!!! Consider level sets of this function:

$$a = 9 - (x + y)^2 \Rightarrow x + y = \pm\sqrt{9 - a}$$

These are straight lines!!! Also, notice that the $\pm$ means that for every value of a, we get two level sets which are reflections of each other across the line $y = -x$. This one looks like



Let's return to our "hill", and calculate the equation to the tangent plane to the hill at the point $(2, 1)$. OK! First, we will need

$$\vec{\nabla} f = \langle -2x, -2y \rangle$$

so

$$\vec{\nabla} f(2, 1) = \langle -4, -2 \rangle$$

And $f(2, 1) = 4$, so we know

$$z - 4 = -4(x - 2) + -2(y - 1)$$

describes the tangent plane, and

$$\tilde{f}(x, y) = -4x - 2y + 14$$

is a linear approximation to f at $(2, 1)$. I wonder what $f(2.5, 0.5)$ might be. An approximation to that is

$$\tilde{f}(2.5, 0.5) = -10 - 1 + 14 = 3$$

How close were we?

$$E = |f(2.5, 3.5) - \tilde{f}(2.5, 0.5)| = |2.5 - 3| = 0.5$$

Now, let's take a look at a tangent plane to g at that same point. First,

$$\vec{\nabla} g = \langle -2(x + y), -2(x + y) \rangle$$

(notice that this is always parallel to $\langle 1, 1 \rangle$. Take a look at the level sets we drew and figure our why)

$$\vec{\nabla} g(2, 1) = \langle -6, -6 \rangle$$

and $g(2,1) = 0$, so

$$z = -6(x-2) - 6(y-2)$$

Let's think now about climbing down our hill. We could go straight down, say along the x axis. Or, we could go in a spiral around the hill. One spiral we could go in is

$$\vec{r}_1(t) = \langle \frac{3t}{4\pi} \cos t, \frac{3t}{4\pi} \sin t \rangle$$

Then our velocity (horizontally) is

$$\vec{r}_1' = \frac{3}{4\pi} \langle \cos t - t \sin t, \sin t + t \cos t \rangle$$

and so our speed (again, in the horizontal directions) is

$$\|\vec{r}_1'(t)\| = \frac{3}{4\pi} \sqrt{(\cos t - t \sin t)^2 + (\sin t + t \cos t)^2} = \frac{3}{4\pi} \sqrt{1 + t^2}$$

Let's compare that with a straight route down of similar speed:

$$\vec{r}_2(t) = \langle \frac{3}{8\pi} t^2, 0 \rangle$$

which has:

$$\vec{r}_2' = \langle \frac{3}{4\pi} t, 0 \rangle$$

and

$$\|\vec{r}_2'\| = \frac{3}{4\pi} t$$

(so similar but not quite the same for small t in speed to $\vec{r}_1(t)$. Now let's compare the change in altitude (f) as a function of time for each path. For $\vec{r}_1(t)$,

$$\frac{df(\vec{r}_1(t))}{dt} = \vec{\nabla} f(\vec{r}_1(t)) \cdot \vec{r}_1'(t) = \langle -2 \left(\frac{3t}{4\pi} \right) \cos t, -2 \left(\frac{3t}{4\pi} \right) \sin t \rangle \cdot \frac{3}{4\pi} \langle \cos t - t \sin t, \sin t + t \cos t \rangle = \left(\frac{-3t}{2\pi} \right)^2$$

Compare this with

$$\frac{df(\vec{r}_2(t))}{dt} = \vec{\nabla} f(\vec{r}_2(t)) \cdot \vec{r}_2'(t) = \langle -2 \frac{3t^2}{8\pi}, 0 \rangle \cdot \langle \frac{3}{4\pi} t, 0 \rangle = - \left(\frac{3}{4\pi} \right)^2 t^3$$

SANITY CHECK: Straight down is the steeper path.