LIST OF TOPICS TO KNOW

• Vectors:
  – Adding, subtracting, and scalar multiplication
  – cross product & dot product
    * computation
    * geometry (ANGLES!!!)
  – Equations of lines and planes
  – Distance between objects in space (planes, lines, points)

• Parametric curves
  – parametrizing curves
  – velocity, acceleration (vector, length)
  – unit tangent (computation, geometry)
  – curvature (vector, length)
  – arc length

• Functions of multiple variables
  – Visualization of the function
    * cross sections
    * factoring/completing the square (may be helpful)
  – level sets
    * drawing simple level sets
    * matching level sets with curves
  – partial derivatives (the gradient vector)
  – linear approximation/tangent plane
  – the chain rule in two dimensions
VECTORS

$\vec{a} = (4, 5, 1) \quad \vec{b} = (3, 2, 1)$

The dot product $\vec{a} \cdot \vec{b}$ is

$$a \cdot b = 4 \cdot 3 + 5 \cdot 2 + 1 \cdot 1 = 12 + 10 + 1 = 23$$

and can also be computed

$$\vec{a} \cdot \vec{b} = ||\vec{a}||||\vec{b}|| \cos(\theta)$$

where $\theta$ is the angle between $\vec{a}$ and $\vec{b}$. Here we don’t know $\theta$ but we can use the dot product to calculate it.

$$||\vec{a}||||\vec{b}|| \cos(\theta) = (42)^{1/2}(15)^{1/2} \cos(\theta) = 23$$

so

$$\theta = \arccos \left( \frac{23}{\sqrt{588}} \right) \approx 0.322 \text{ radians}$$

The cross product is

$$\vec{a} \times \vec{b} = \begin{vmatrix}
\hat{i} & \hat{j} & \hat{k} \\
4 & 5 & 1 \\
3 & 2 & 1
\end{vmatrix} = 3\hat{i} - \hat{j} - 7\hat{k}$$

We could also calculate the length of cross product with the formula

$$||\vec{a} \times \vec{b}|| = ||\vec{a}||||\vec{b}|| \sin(\theta) \approx \sqrt{588} \sin(0.322) \approx 7.67$$

So, find the area of the parallelogram with sides $\vec{a}$ and $\vec{b}$. We already have.

What is the equation of the plane containing the points $(1, 1, 1)$, $(3, 3, 3)$ and $(1, 2, 3)$?

We need two vectors that lie in the plane. Two possibilities are $(2, 2, 2)$ and $(0, 1, 2)$. Then the normal to the plane is

$$\langle 2, 2, 2 \rangle \times \langle 0, 1, 2 \rangle = \begin{vmatrix}
\hat{i} & \hat{j} & \hat{k} \\
2 & 2 & 2 \\
0 & 1 & 2
\end{vmatrix} = 2\hat{i} - 4\hat{j} + 2\hat{k}$$

SANITY CHECK!!!! Is this perpendicular to my vectors?? (Use dot product to show that it is).

My plane is given by

$$2(x - 1) - 4(y - 1) + 2(z - 1) = 0 \Rightarrow 2x - 4y + 2z = 0$$

How far is the point $(2, 0, 1)$ from the plane?

First, let’s write a vector from the plane to the point:

$$\vec{v} = \langle 2 - 1, 0 - 1, 1 - 1 \rangle = \langle 1, -1, 0 \rangle$$

We know that

$$\cos(\theta) = \frac{\vec{v} \cdot \vec{n}}{||\vec{v}||||\vec{n}||}$$
and we can draw some triangles to see that the distance we are looking for, \( d \), is
\[
d = \| \| \vec{v} \| \cos(\theta) \|
\]
so
\[
d = \left| \frac{\vec{v} \cdot \vec{n}}{\| \vec{v} \| \| \vec{n} \|} \right| = \frac{6}{\sqrt{24}}
\]

**PARAMETRIC CURVES**

Can we define a parametric curve that is an inward, clockwise spiral that ends up at the point \((1, 1)\)?

**OF COURSE WE CAN!!!** If we wanted to parametrise a “clockwise” oriented circle, we could use
\[
\vec{r}(t) = \langle R \sin t, R \cos t \rangle
\]
(this is different from what we normally write down for a circle). Now, we simply need the radius to decrease to 0 as \( t \to \infty \) and increase to \( \infty \) as \( t \to -\infty \). So we are after \( R(t) \) so that \( \vec{r}(t) = \langle R(t) \sin t, R(t) \cos t \rangle \) is a spiral. We can use \( R(t) = e^{-t} \).

**NOTICE** - there are other possible choices that would give us a similar looking spiral - these MAY OR MAY NOT be exactly the same spiral - what if I chose \( R(t) = \frac{1}{t} \), or \( R(t) = e^{-3t} \)????

Our curve is going to be \( \vec{r}(t) = \langle e^{-t} \sin t, e^{-t} \cos t \rangle \). We can now ask a lot of questions about the curve.

What is the “velocity” of the curve with respect to \( t \)?
\[
\vec{r}'(t) = \langle -e^{-t} \sin t + e^{-t} \cos t, -e^{-t} \cos t - e^{-t} \sin t \rangle
\]
and the “speed”?
\[
\| \vec{r}'(t) \| = \sqrt{e^{-2t}(\cos t - \sin t)^2 + e^{-2t}(\cos t + \sin t)^2} = e^{-t}\sqrt{2}
\]

And the tangent direction? (Remember this is the DIRECTION of the rate of change but ignores speed - i.e. it is a unit vector)
\[
\vec{T}(t) = \frac{\vec{r}'(t)}{\| \vec{r}'(t) \|} = \frac{1}{e^{-t}\sqrt{2}} \langle -e^{-t} \sin t + e^{-t} \cos t, -e^{-t} \cos t - e^{-t} \sin t \rangle = \frac{-1}{\sqrt{2}} \langle \sin t - \cos t, \cos t + \sin t \rangle
\]

What about the curvature, \( \vec{\kappa} \)?
\[
\vec{\kappa} = \frac{\vec{T}'(t)}{\| \vec{T}'(t) \|} = \frac{-1}{e^{-t}2} \langle \cos t + \sin t, \cos t - \sin t \rangle
\]
and let’s calculated the length of \( \vec{\kappa} \)
\[
\| \vec{\kappa} \| = \left( \frac{1}{2e^{-t}} \right) \sqrt{(\cos t + \sin t)^2 + (\cos t - \sin t)^2} = \frac{\sqrt{2}}{2e^{-t}}
\]
Finally, let’s calculate the arc length from where we pass through $(1, 2)$ to when $t = a$. Notice that $\vec{r}(0) = \langle 1, 2 \rangle$, so we want

$$\int_0^a \|\vec{r}'(t)\| dt = \int_0^a \sqrt{2e^{-t}} dt = -\sqrt{2}e^{-t}\Big|_0^a = \sqrt{2}(1 - e^{-a})$$

And if we want the length of our spiral from $(1, 2)$ to $(1, 1)$, we see that as $a \to \infty$, we get it. That length is then $\lim_{a \to \infty} \sqrt{2}(1 - e^{-a}) = \sqrt{2}$.

**FUNCTIONS OF MULTIPLE VARIABLES**

Can we visualize the following function?

$$f(x, y) = 10 - (x^2 + y^2)$$

We might consider inspecting the VERTICAL planes $x = 0$ and $y = 0$ to see that vertical cross sections of this are parabolas, and conclude that this is something like an upside down bowl. What do its level sets look like??

$$a = 9 - (x^2 + y^2) \Rightarrow x^2 + y^2 = 10 - a$$

so they are circles of radius $\sqrt{9-a}$. If $a > 9$, we don’t have any level sets - that is not in the range of the function.

GREAT! What about

$$g(x, y) = 9 - (x + y)^2$$

Let’s try the same type of tricks. If we “freeze” our variables, we see that vertical cross sections on the $x, z$ and $y, z$ planes are the same as the first example. HOWEVER!!!!!!! Consider level sets of this function:

$$a = 9 - (x + y)^2 \Rightarrow x + y = \pm \sqrt{9 - a}$$

These are straight lines!!! Also, notice that the $\pm$ means that for every value of $a$, we get two level sets which are reflections of each other across the line $y = -x$. This one looks like
Let’s return to our “hill”, and calculate the equation to the tangent plane to the hill at the point \((2, 1)\). OK! First, we will need

\[
\vec{\nabla} f = \langle -2x, -2y \rangle
\]

so

\[
\vec{\nabla} f(2, 1) = \langle -4, -2 \rangle
\]

And \(f(2, 1) = 4\), so we know

\[
z - 4 = -4(x - 2) - 2(y - 1)
\]

describes the tangent plane, and

\[
\tilde{f}(x, y) = -4x - 2y + 14
\]

is a linear approximation to \(f\) at \((2, 1)\). I wonder what \(f(2.5, 0.5)\) might be. An approximation to that is

\[
\tilde{f}(2.5, 0.5) = -10 - 1 + 14 = 3
\]

How close were we?

\[
E = |f(2.5, 3.5) - \tilde{f}(2.5, 0.5)| = |2.5 - 3| = 0.5
\]

Now, let’s take a look at a tangent plane to \(g\) at that same point. First,

\[
\vec{\nabla} g = \langle -2(x + y), -2(x + y) \rangle
\]

(notice that this is always parallel to \((1, 1)\). Take a look at the level sets we drew and figure our why)

\[
\vec{\nabla} g(2, 1) = \langle -6, -6 \rangle
\]
and \(g(2, 1) = 0\), so
\[
z = -6(x - 2) - 6(y - 2)
\]

Let’s think now about climbing down our hill. We could go straight down, say along the \(x\) axis. Or, we could go in a spiral around the hill. One spiral we could go in is
\[
\vec{r}_1(t) = \left(\frac{3t}{4\pi} \cos t, \frac{3t}{4\pi} \sin t\right)
\]

Then our velocity (horizontally) is
\[
\vec{r}'_1 = \frac{3}{4\pi} \langle \cos t - t \sin t, \sin t + t \cos t \rangle
\]

and so our speed (again, in the horizontal directions) is
\[
\|\vec{r}'_1(t)\| = \frac{3}{4\pi} \sqrt{(\cos t - t \sin t)^2 + (\sin t + t \cos t)^2} = \frac{3}{4\pi} \sqrt{1 + t^2}
\]

Let’s compare that with a straight route down of similar speed:
\[
\vec{r}_2(t) = \left(\frac{3}{8\pi} t^2, 0\right)
\]

which has:
\[
\vec{r}'_2 = \langle \frac{3}{4\pi} t, 0 \rangle
\]

and
\[
\|\vec{r}'_2\| = \frac{3}{4\pi} t
\]

(so similar but not quite the same for small \(t\) in speed to \(\vec{r}_1(t)\)). Now let’s compare the change in altitude \((f)\) as a function of time for each path. For \(\vec{r}_1(t)\),
\[
\frac{df(\vec{r}_1(t))}{dt} = \nabla f(\vec{r}_1(t)) \cdot \vec{r}'_1(t) = \langle -2 \left(\frac{3t}{4\pi}\right) \cos t, -2 \left(\frac{3t}{4\pi}\right) \sin t \rangle \cdot \frac{3t}{4\pi} \langle \cos t - t \sin t, \sin t + t \cos t \rangle = \left(-\frac{3t}{2\pi}\right)^2
\]

Compare this with
\[
\frac{df(\vec{r}_2(t))}{dt} = \nabla f(\vec{r}_2(t)) \cdot \vec{r}'_2(t) = \langle -2 \left(\frac{3t^2}{8\pi}\right), 0 \rangle \cdot \left(\frac{3}{4\pi}\right) t \langle \frac{3}{4\pi} t, 0 \rangle = -\left(\frac{3}{4\pi}\right)^2 t^3
\]

SANITY CHECK: Straight down is the steeper path.