Math 234 Exam 3 Review Sheet

Jim Brunner

LIST OF TOPICS TO KNOW

- Vector Fields
  - Clairaut's Theorem & Conservative Vector Fields
  - Curl
  - Divergence
- Area & Volume Integrals
  - Using Coordinate Transforms
  - Changing the Order of Integration
- Line Integrals
  - Scalar Fields & Arc Length Integrals
  - Work Integrals
    - Direct Computation
    - Fundamental Theorem for Conservative Vector Fields
    - Using Green's (Stokes') Theorem
  - Flux Integrals
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    - Using Gauss's (Divergence) Theorem
- Parametric Surfaces
  - Normal Vectors
  - Scalar Integrals & Surface Area
  - Flux Integrals
    - Direct Computation
    - Gauss's (Divergence) Theorem

Vector Fields

A vector field is a "vector valued" function - it is a function of two (or three) variables which takes vector values:

$$\vec{F}(x,y) = \langle P(x,y), Q(x,y)\rangle \quad or \quad \vec{F}(x,y,z) = \langle P(x,y,z), Q(x,y,z), R(x,y,z)\rangle$$

Conservative Vector Fields

Our first encounter with vector fields was when we computed the gradient of a function of two (or three) variables, $f(x,y)$. In that case

$$P(x,y) = \frac{\partial f}{\partial x} \quad Q(x,y) = \frac{\partial f}{\partial y}$$

Vector fields which are the gradient of a function are called *conservative.* Precisely:

Definition. *If, for a vector field $\vec{F}(x,y) = \langle P(x,y), Q(x,y)\rangle$, there exists a smooth function $f(x,y)$ such that*

$$\nabla f(x,y) = \vec{F}(x,y)$$

then $\vec{F}$ is conservative.

We can check to see if a vector field is conservative using Clairaut's theorem. Simply put, we are comparing the mixed partial derivatives of a hypothetical $f(x,y)$ such that $\nabla f = \vec{F}$. If this f exists and is smooth, then its mixed partials must be equal. We can compute these hypothetical mixed partials. If $\nabla f = \vec{F}$, then

$$\frac{\partial P}{\partial y} = f_{xy} \quad \frac{\partial Q}{\partial x} = f_{yx}$$

Therefore, if we have a conservative vector field,

$$\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}$$

It turns out that for a simply connected region (such as the whole plane, or a disk) it is true that if that is true for the whole region, then $\vec{F}$ is conservative. In other words

$$\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x} \Leftrightarrow \vec{F} \text{ is conservative}$$

If $\vec{F}$ is conservative, we can figure out what f is. We know that

$$f_x = P(x,y) \quad \& \quad f_y = Q(x,y)$$

And we can use this information to figure out f by anti-differentiating. For example, consider

$$\vec{F} = \langle 2x + y\cos(x), \sin(x) + 3y^2\rangle$$

We see that

$$f_x = 2x + y\cos(x)$$

and we know that

$$f(x,y) = \int f_x dx + g(y)$$

So,

$$f(x,y) = \int (2x + y\cos(x))\, dx + g(y) = x^2 + y\sin(x) + g(y)$$

To figure out what $g(y)$ is, we need to use f_y:

$$f_y = \frac{\partial}{\partial y}\left(x^2 + y\sin(x) + g(y)\right)$$

and so

$$\sin(x) + 3y^2 = \sin(x) + g'(y) \Rightarrow g'(y) = 3y^2 \Rightarrow g(y) = y^3 + C$$

We don't really need the constant, we care about the vector field and its properties. We have then that

$$f(x,y) = x^2 + y\sin(x) + y^3$$

Curl

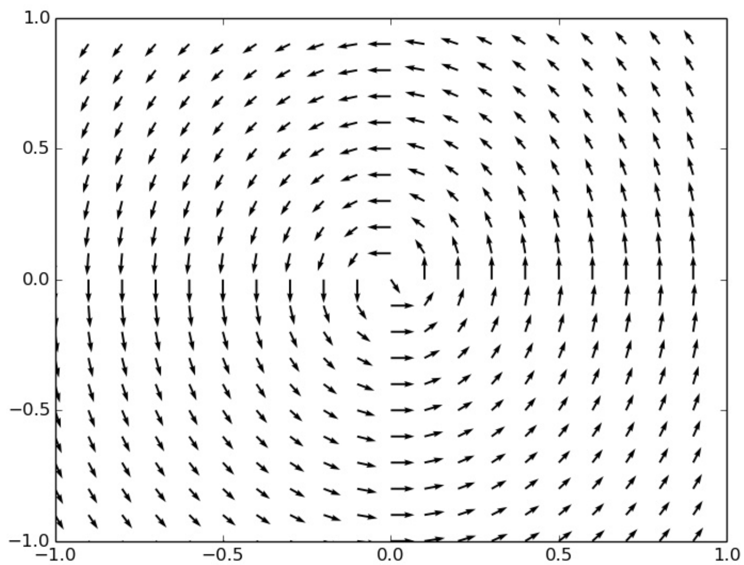
Looking back at the test we used to determine if a vector field is conservative, we can rewrite the condition as

$$\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} = 0$$

The quantity

$$\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}$$

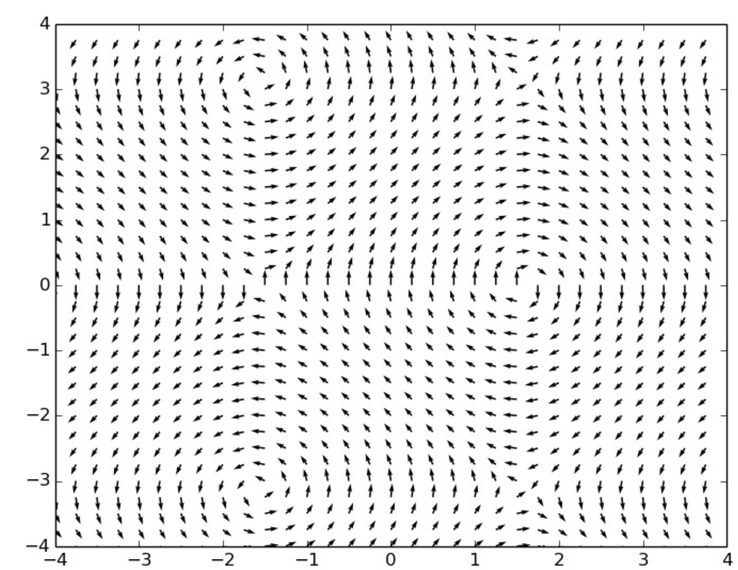
is called the *curl* of $\vec{F}$, denoted $curl(\vec{F})$ or $\nabla \times \vec{F}$. In three dimensions, curl is a vector. In two dimensions, it is scalar, but can be thought of as a vector by thinking of the two dimensional space as existing in three dimensions. Physically, we can think of curl as rotation. It can be helpful to imagine a paddle wheel in the vector field. If it is spinning counter clockwise, the vector field has positive curl at that point.
The vector field $\vec{F} = \langle -2y, 3x\rangle$ looks like:



this has $\nabla \times \vec{F} = 5$ everywhere. On the other hand, if

$$\vec{G} = \langle \sin(y), \cos(x) \rangle$$

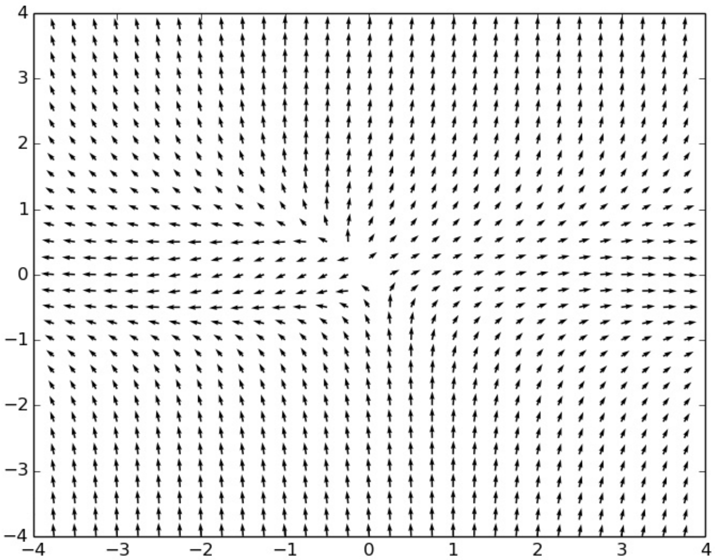
then $\nabla \times \vec{G} = -\sin(x) - \cos(y)$, which is not constant. This vector field looks like:



At the point $\left(\frac{\pi}{2}, \frac{\pi}{2}\right)$, this has negative curl. At $\left(-\frac{\pi}{2}, \pi\right)$, this has positive curl. We already have noticed that a conservative vector field has no curl. For example, our conservative vector field

$$\vec{F} = \langle 2x + y\cos(x), \sin(x) + 3y^2 \rangle$$

looks like



Any paddle wheel we insert in this vector field is not going to spin.

Divergence

The conservative vector may not have curl, but it does have another property that can be determine just from looking at the plot. That is *divergence*. Physically, divergence can be interpreted in terms of compressibility. Imagine a blob of (non-diffusive) food colouring placed into a thin layer of fluid whose motion is described by the vector field. If the blob gets bigger - the food colouring is spread apart - the vector field has positive divergence. If it gets smaller - the food colouring is compressed - the vector field has negative divergence. Alternatively, we can think about flux of fluid through a closed loop. Positive divergence will mean fluid *leaving* the area enclosed by the loop, and so will correspond to positive flux. We will revisit that idea later in the *Divergence Theorem* for flux integrals.
The vector field we have a picture of looks like it has positive divergence everywhere. We can calculate divergence with the formula

$$div(\vec{F}) = \nabla \cdot \vec{F} = \frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y}$$

in two dimensions, or

$$div(\vec{F}) = \nabla \cdot \vec{F} = \frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} + \frac{\partial R}{\partial z}$$

in three dimensions. For our vector field,

$$\nabla \cdot \vec{F} = 2 - y\sin(x) + 6y = 2 + y(6 - \sin(x)) > 0$$

so our vector field has positive divergence everywhere. Note that there can certainly be vector fields with positive divergence at some points and negative divergence at others.

Area & Volume Integrals

Area and volume integrals, also known as double and triple integrals, can be evaluated simply by iteratively evaluating individual integrals, working from the inside out. The outer most integration variable should have constant bounds of integration for evaluation of a definite integral.

Using a Coordinate Transform

In some cases, we may make our lives easier by a change of variables. The most common change of variables we make is changing from Cartesian to polar coordinates, so that is a good example to start with. If we are trying to integrate over a circle of radius R, we may let

$$x = r\cos(\theta) \quad y = r\sin(\theta)$$

Our circle is the image of a simple rectangle in (r, θ) coordinates. Put another way, the domain that becomes our circle when we plug it into the transform above is simply the rectangle $0 \leq r \leq R$, $0 \leq \theta \leq 2\pi$. This means our integral has constant bounds, and so is easier. When we do this, however, we have to correct for area changes under the transformation. We do this by multiplying the function we want to integrate by the determinant of the Jacobian of the transformation, in this case r. For example:

$$\int_{-2}^{2}\int_{-\sqrt{4-y^2}}^{\sqrt{4-y^2}} e^{x^2+y^2}dxdy$$

is impossible to integrate! Not only is our domain a circle, but the function itself isn't one we can integrate. However, using $x = r\cos(\theta)$, $y = r\sin(\theta)$, we have

$$\begin{aligned}\int_0^2\int_0^{2\pi} re^{r^2}d\theta dr &= 2\pi\int_0^2 re^{r^2}dr \\ &= \pi\int_0^4 e^u du \\ &= \pi\left(e^4 - 1\right)\end{aligned}$$

We might use other coordinate transforms, of course. In general, faced with

$$\iint\limits_{(x,y)\in D} f(x,y)dA$$

we can use a transformation (if it helps):

$$x = h(u,v) \quad y = g(u,v)$$

and we compute the determinant of the Jacobian:

$$|J| = \begin{vmatrix}\frac{\partial h}{\partial u} & \frac{\partial g}{\partial u} \\ \frac{\partial h}{\partial v} & \frac{\partial g}{\partial v}\end{vmatrix}$$

and then

$$\iint\limits_{(x,y)\in D} f(x,y)dA = \iint\limits_{(u,v)\in D^*} f(h(u,v),g(u,v))|J|dA$$

Where D is the image of D^* under this transformation (meaning if I plugged all of the infinitely many points inside D^* into h and g and plotted them, I would end up with a picture of D). For example, consider

$$\int_0^1\int_0^{\sqrt{x}} \frac{y}{\sqrt{x}}dydx$$

We can compute this directly, of course, but let's consider using the transformation

$$x = u^2 \quad y = uv$$

Then we have the bounds

$$0 \le u^2 \le 1 \quad 0 \le uv \le u \Rightarrow 0 \le u \le 1 \quad 0 \le v \le 1$$

and the Jacobian

$$|J| = \begin{vmatrix} 2u & 0 \\ v & u \end{vmatrix} = 2u^2$$

so the integral is

$$\int_0^1\int_0^1 2u^2vdudv = \frac{2}{3}\int_0^1 vdv = \frac{1}{3}$$

Switching the Order of Integration

It may be impossible, or at least difficult, to evaluate the individual inner integral. Furthermore, in the case of a volume integral, after evaluating the inner integral the second could be impossible to evaluate. In this case, it might be helpful to integrate in a different order. Unfortunately this is not as simple as just switching the order of the integration symbols! In an area or volume integral which is not over a rectangle, the inner bounds are not constant. The are expressed as functions of the outer variable (or variables). If we switch the order of integration, we must find a new way to express the domain of integration so that we have constant bounds on the outside. For example, to evaluate the integral

$$\int_0^8\int_{y^{\frac{1}{3}}}^2 \left(x^4+1\right)^{\frac{1}{2}} dxdy$$

we need to switch the order of integration. Our new integral will look like

$$\int_a^b\int_{f(x)}^{g(x)} \left(x^4+1\right)^{\frac{1}{2}} dydx$$

where $a \le x \le b$, and a and b are constant, while $f(x) \le y \le g(x)$, and $f(x)$ and $g(x)$ are allowed to be functions of x. Finally we need:

$$\left\{ (x,y) \,\middle|\, \begin{array}{c} a \le x \le b \\ f(x) \le y \le g(x) \end{array} \right\} = \left\{ (x,y) \,\middle|\, \begin{array}{c} 0 \le y \le 8 \\ y^{\frac{1}{3}} \le x \le 2 \end{array} \right\}$$

In other words, these must describe the same domain. In this case, we see that

$$a = 0 \quad b = 2 \quad f(x) = 0 \quad g(x) = x^3$$

so our integral becomes

$$\int_0^2 \int_0^{x^3} \left(x^4 + 1\right)^{\frac{1}{2}} dydx$$

and we can evaluate this:

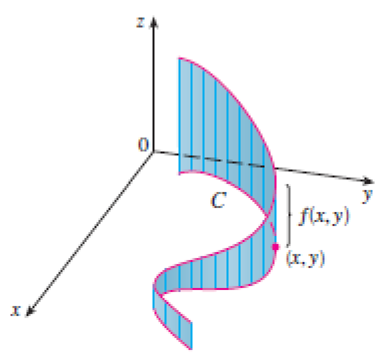
$$\begin{aligned} \int_0^2 \int_0^{x^3} \left(x^4 + 1\right)^{\frac{1}{2}} dydx &= \int_0^2 x^3 \left(x^4 + 1\right)^{\frac{1}{2}} dx \\ &= \int_1^{17} u^{\frac{1}{2}} du \\ &= \frac{1}{6}\left(17^{\frac{3}{2}} - 1\right) \end{aligned}$$

Line Integrals

Line integrals involve calculating the integral of some quantity (work, flux, or some scalar function) along some curve in 2 or 3 dimensional space. Often we will need to know how to parametrise the curve. There are three major types of line integrals: scalar, work, and flux.

Line Integrals Over Scalar Fields

The simplest is a line integral of some scalar function over a curve. The picture to have in mind is



For this type of integral, we need the parametrisation of our curve. If $\mathcal{C}$ is the curve parametrised by $\vec{r}(t)$, $a \leq t \leq b$, then

$$\int_{\mathcal{C}} f(x,y)ds = \int_a^b f(\vec{r}(t))|r'(t)|dt$$

If integrate the scalar function $f(x,y) = 1$, we have the length of the curve. For example, we might want to compute

$$\int_{\mathcal{C}} xy^4 ds$$

where $\mathcal{C}$ is the half circle in the right half plane with radius 5. We parametrise $\mathcal{C}$ by

$$r(t) = \langle 5\cos(t), 5\sin(t)\rangle \quad -\frac{\pi}{2} \leq t \leq \frac{\pi}{2}$$

then we compute

$$|r'(t)| = |\langle -5\sin(t), 5\cos(t)\rangle| = 5$$

so the integral is

$$\begin{aligned}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} 5\cos(t)\left(5\sin(t)\right)^4(5)dt &= 5^6 \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos(t)\sin^4(t)dt \\ &= 5^5 \sin^5(t)\Big|_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \\ &= 2(5^5)\end{aligned}$$

Work Integrals

A line integral over a vector field is known as a work integral. This type of integral tells us something about how the vector field "pushes" along a curve.

Direct Computation

We can calculate a work integral by parametrising our curve $\mathcal{C}$ as $\vec{r}(t)$, $a \leq t \leq b$, and using the formula

$$\int_{\mathcal{C}} \vec{F} \cdot d\vec{r} = \int_{\mathcal{C}} \vec{F} \cdot \vec{T} ds = \int_a^b \vec{F}(\vec{r}(t)) \cdot \vec{r}'(t)dt$$

Here, $\vec{T} = \frac{r'(t)}{|r'(t)|}$ is the tangent vector, and $ds = |r'(t)|dt$ as before. For example, we might want to know the work done by the vector field

$$\vec{F} = \langle -x, y\rangle$$

on the curve

$$\vec{r}(t) = \langle t, t^2\rangle \quad -1 \leq t \leq 1$$

We calculate

$$\vec{r}'(t) = \langle 1, 2t\rangle$$

and apply the formula to see

$$\begin{aligned}\int_{\mathbb{C}} \vec{F}\cdot d\vec{r} &= \int_{-1}^{1} \langle -t, t^2\rangle \cdot \langle 1, 2t\rangle dt \\ &= \int_{-1}^{1} (2t^3 - t)dt \\ &= \left(\frac{t^4}{2} - \frac{t^2}{2}\right)\bigg|_{-1}^{1} \\ &= 0\end{aligned}$$

The Fundamental Theorem for Conservative Vector Fields

Work integrals are actually much easier when the vector field is conservative. If $\vec{F}$ is conservative, we know that there is an $f(x, y)$ such that

$$\vec{F} = \nabla f$$

If this is the case, we can calculate the integral line integral over the curve $\mathcal{C}$ which starts at the point (x_0, y_0) and ends at the point (x_1, y_1) with the formula

$$\int_{\mathcal{C}} \vec{F}\cdot d\vec{r} = f(x_1, y_1) - f(x_0, y_0)$$

Notice, this means that the work integral over a conservative vector field only depends on the endpoints of the curve. Let's revisit the conservative vector field we looked at earlier:

$$\vec{F} = \langle 2x + y\cos(x), \sin(x) + 3y^2\rangle$$

We figured out that if

$$f(x, y) = x^2 + y\sin(x) + y^3$$

then $\vec{F} = \nabla f$. We can then easily compute

$$\int_{\mathcal{C}} \langle 2x + y\cos(x), \sin(x) + 3y^2\rangle \cdot d\vec{r}$$

where $\mathcal{C}$ is the spiral from the origin to the point $(-\pi, 0)$, which is parametrised

$$\vec{r}(t) = \langle t\cos(t), t\sin(t)\rangle \quad 0 \le t \le \pi$$

In fact, we don't really care about how to parametrise this curve, or even what the curve is. We only need to know it starts at $(0, 0)$ and ends at $(-\pi, 0)$. So,

$$\int_{\mathcal{C}} \langle 2x + y\cos(x), \sin(x) + 3y^2\rangle \cdot d\vec{r} = f(-\pi, 0) - f(0, 0) = \pi^2$$

Notice, that if the curve is closed, we don't have a starting and ending point. However, if that is the case, we can pick any point on our curve to be both the start and end point. Then, it is easy to see the integral is 0. So, for any closed curve $\mathcal{C}$ and conservative vector field $\vec{F}$, we have

$$\oint_{\mathcal{C}} \vec{F}\cdot d\vec{r} = 0$$

Using Green's (Stokes') Theorem

If we have a closed curve oriented counter clockwise which encloses the region $\mathcal{D}$, then

$$\oint_{\mathcal{C}} \vec{F} \cdot d\vec{r} = \iint_{\mathcal{D}} \nabla \times \vec{F} dA$$

This is called *Green's Theorem* or *Stokes' Theorem*. We have already seen the quantity

$$curl(\vec{F}) = \nabla \times \vec{F} = \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}$$

and now we see we have a very important use for it. This theorem matches with what we said about conservative vector fields because, as we have seen already, the curl of a conservative vector field is always 0. If we have a curve that is oriented clockwise, we need to multiply the right hand side of this formula by -1.
This theorem is especially useful if $\mathcal{C}$ is not a smooth curve. For example,

$$\oint_{\mathcal{C}} \langle \sin(y), \cos(x) \rangle \cdot d\vec{r}$$

where $\mathcal{C}$ is the triangle that connects the points $(0, 1)$, $(-1, 0)$ and $(0, -1)$, oriented counter clockwise. If we were to integrate this using a direct computation, we would need to compute three different line integrals and add them together. However, using Green's theorem (and the curl we computed earlier)

$$\begin{aligned}
\int_{\mathcal{C}} \langle \sin(y), \cos(x) \rangle \cdot d\vec{r} &= \int_{-1}^{0} \int_{-x-1}^{x+1} -(\sin(x) + \cos(y))\, dy dx \\
&= -\int_{-1}^{0} (\sin(x+1) - \sin(-x-1) + 2(x+1)\sin(x))\, dx \\
&= -\int_{-1}^{0} 2\sin(x+1) + 2\sin(x) + 2x\sin(x) dx \\
&= 2\left(\cos(x+1) + \cos(x)\right)|_{-1}^{0} - 2\left(\cos(1) + \sin(x)|_{-1}^{0}\right) \\
&= 2\cos(1) + 2\sin(1)
\end{aligned}$$

Flux Integrals

We already mentioned the word flux. It is a quantity that tells us something about how a vector field moves "through" a curve. We may, for example, want to know the flux of the velocity field of the Colorado river through the Hoover Dam.

Direct Computation

To calculate the *total* flux of a vector field through a curve $\mathcal{C}$, we can calculate the line integral

$$\int_{\mathcal{C}} \vec{F} \cdot \hat{n} ds$$

where $\hat{n}$ is the normal vector to the curve. This matches our intuition because it tells us immediately that if the field points along $\mathcal{C}$, there will be no flux through $\mathcal{C}$, and flux will be highest when the vector field points perpendicular to $\mathcal{C}$. We can compute this in much the same way that we computed the line integral of a scalar function. To do this, we find a parametrisation $\vec{r}(t)$, $a \leq t \leq b$ for $\mathcal{C}$, and then find the unit normal to $\vec{r}(t)$, $\hat{n}(t)$. Then,

$$\int_{\mathcal{C}} \vec{F} \cdot \hat{n} ds = \int_a^b \vec{F}(\vec{r}(t)) \cdot \hat{n}(t) |\vec{r}'(t)| dt$$

For example, we can calculate the total flux of the vector field

$$\vec{F} = \langle -x, y \rangle$$

through the curve $r(t) = \langle t, t^2 \rangle$, $-1 \leq t \leq 1$, which is part of a parabola. We compute

$$r'(t) = \langle 1, 2t \rangle$$

and so

$$\hat{n} = \frac{1}{\sqrt{1+4t^2}} \langle -2t, 1 \rangle$$

and

$$|r'(t)| = \sqrt{1+4t^2}$$

Then our flux is

$$\begin{aligned} \int_{\mathcal{C}} \langle -x, y \rangle \cdot \hat{n} ds &= \int_{-1}^{1} \frac{1}{\sqrt{1+4t^2}} \langle -t, t^2 \rangle \cdot \langle -2t, 1 \rangle \sqrt{1+4t^2} dt \\ &= \int_{-1}^{1} 3t^2 dt \\ &= t^3 |_{-1}^{1} \\ &= 2 \end{aligned}$$

Using Gauss's (Divergence) Theorem

We might also care about flux through a closed curve. In that case, we have a relationship between the flux through the curve and the divergence of the vector field. That is called the *Divergence Theorem* or *Gauss's Theorem*. It states that

$$\oint_{\mathcal{C}} \vec{F} \cdot \hat{n} ds = \iint_{\mathcal{D}} \nabla \cdot \vec{F} dA$$

where $\mathcal{D}$ is the area enclosed by $\mathcal{C}$. For example, with our vector field from before

$$\vec{F} = \langle 2x + y\cos(x), \sin(x) + 3y^2 \rangle$$

we can compute the total flux through the square centred at the origin with side lengths 2.

$$\begin{aligned} \oint_{\mathcal{C}} \vec{F} \cdot \hat{n} ds &= \int_{-1}^{1} \int_{-1}^{1} 2 + y(6 - \sin(x)) dy dx \\ &= \int_{-1}^{1} 4 dx \\ &= 8 \end{aligned}$$

Parametric Surfaces

We can write down a parametrisation for a two dimensional surface that exists in three dimensional space in much the same way that we could parametrise a one dimensional curve in two and three dimensional space. The three surfaces we have seen the most are

- A sphere of radius ρ:

$$\vec{r}(\theta, \phi) = \langle \rho\cos(\theta)\sin(\phi), \rho\sin(\theta)\sin(\phi), \cos(\phi)\rangle \quad 0 \le \theta \le 2\pi \quad 0 \le \phi \le \pi$$

- A cylinder or radius ρ, height H, and base on the xy plane:

$$\vec{r}(\theta, z) = \langle \rho\cos(\theta), \rho\sin(\theta), z\rangle \quad 0 \le \theta \le 2\pi \quad 0 \le z \le H$$

- The graph of a function $f(x, y)$ over some domain $\mathcal{D}$:

$$\vec{r}(u, v) = \langle u, v, f(u, v)\rangle \quad (u, v) \in \mathcal{D}$$

Normal Vectors

The normal vector a surface is the unit vector pointing perpendicular to the surface. In the case of the graph of a function, this is exactly the same as the normal vector to the tangent plane at a point. In fact, we could use this normal to find a tangent plane to any parametric surface at some point. To calculate the normal vector to a surface $\mathcal{S}$ parametrised by $\vec{r}(u, v)$, we use the formula

$$\hat{n} = \frac{\vec{r}_u \times \vec{r}_v}{|\vec{r}_u \times \vec{r}_v|}$$

where

$$\vec{r}_u = \langle \frac{\partial r_1}{\partial u}, \frac{\partial r_2}{\partial u}\frac{\partial r_3}{\partial u}\rangle$$

and

$$\vec{r}_v = \langle \frac{\partial r_1}{\partial v}, \frac{\partial r_2}{\partial v}\frac{\partial r_3}{\partial v}\rangle$$

We should, if our surface is closed, be sure to take the outward pointing normal. For example, we can compute the normal to the sphere of radius ρ. We have

$$\vec{r}_\theta = \langle -\rho\sin(\theta)\sin(\phi), \rho\cos(\theta)\sin(\phi), 0\rangle$$

and

$$\vec{r}_\phi = \langle \rho\cos(\theta)\cos(\phi), \rho\sin(\theta)\cos(\phi), -\rho\sin(\theta)\rangle$$

and so their cross product is

$$\vec{r}_\phi \times \vec{r}_\theta = \langle \rho^2\cos(\theta)\sin^2(\phi), \rho^2\sin(\theta)\sin^2(\phi), \rho^2\sin(\phi)\cos(\phi)\rangle$$

and so

$$|\vec{r}_\phi \times \vec{r}_\theta| = \rho^2\sin(\phi)$$

and so

$$\hat{n} = \langle \cos(\theta)\sin(\phi), \sin(\theta)\sin(\phi), \cos(\phi)\rangle$$

Surface Integrals of Scalar Functions

We can calculate an integral of a scalar function over a surface $\mathcal{S}$ which is parametrized by $\vec{r}(u,v)$, with $(u,v) \in \mathcal{D}$ with the formula

$$\iint_{\mathcal{S}} f(x,y,z)dS = \iint_{\mathcal{D}} f(\vec{r}(u,v))|\vec{r}_u \times \vec{r}_v|dA$$

For example, we might integrate $f(x,y,z) = x^2+y^2+z^2$ on the sphere of radius 1. We already computed that $|\vec{r}_\phi \times \vec{r}_\theta| = \sin(\phi)$. Furthermore, it is not hard to see that $f(\vec{r}(\theta,\phi)) = 1$. Then we have

$$\begin{aligned}\iint_{\mathcal{S}} f(x,y,z)dS &= \int_0^{2\pi}\int_0^{\pi} \sin(\phi)d\phi d\theta \\ &= \int_0^{2\pi} 2d\theta \\ &= 4\pi\end{aligned}$$

Notice that these integrals tell us the surface area of our surface if we integrate the constant function $f(x,y,z) = 1$.

Flux Integrals

As in two dimensions, we can calculate flux. Now, our flux is flowing through two dimensional surfaces. Physically, the idea is the same.

Direct Computation

To calculate the flux of a vector field $\vec{F}$ through a surface $\mathcal{S}$ parametrised by $\vec{r}(t)$, which has normal vector $\hat{n}$, we have

$$\iint_{\mathcal{S}} \vec{F} \cdot \hat{n} dS = \iint_{\mathcal{D}} \vec{F}(\vec{r}(t)) \cdot (\vec{r}_u \times \vec{r}_v)dA$$

We could calculate the flux of the vector field

$$\vec{F} = \langle 2x+y, z+2y, z+x+y\rangle$$

through a sphere of radius 1. We know the normal vector and parametrization for the sphere. We can plug in and chug through.

Using Gauss's (Divergence) Theorem

Instead of all that, let's recall that in two dimensions, we had a pretty nice theorem if we wanted to calculated the flux through a closed curve. In three dimensions, we have a nearly identical theorem. In fact, it's so similar that it has the same name. In three dimensions, a closed surface will enclose a solid. Then, if $\mathcal{S}$ encloses the solid $\mathcal{E}$, *Gauss's Theorem* is

$$\iint\limits_{\mathcal{S}} \vec{F} \cdot \hat{n} dS = \iiint\limits_{\mathcal{E}} \nabla \cdot \vec{F} dV$$

For our example, we see that

$$\nabla \cdot \vec{F} = 2 + 2 + 1 = 5$$

So the flux through the sphere is just

$$\iint\limits_{\mathcal{S}} \vec{F} \cdot \hat{n} dS = 5 \iiint\limits_{\mathcal{E}} dV$$

or, simply

$$\iint\limits_{\mathcal{S}} \vec{F} \cdot \hat{n} dS = 5 \times (\text{the volume of the sphere})$$

which is

$$\iint\limits_{\mathcal{S}} \vec{F} \cdot \hat{n} dS = 5 \left(\frac{4}{3} \pi \right) = \frac{20\pi}{3}$$