Robust Permanence of Deterministic Reaction Network Models

Jim Brunner & Gheorghe Craciun

University of Wisconsin - Madison
Permanence of Polynomial Dynamical Systems

To determine permanence of polynomial dynamical systems,

\[ \dot{x} = \sum_{i=1}^{n} k_i(t)x^{s_i}v_i \]

we inspect graphs \( G \) which *generate* these dynamical systems. We define for these graphs the condition Tropically Endotactic, which is sufficient to conclude that a dynamical system generated by \( G \) is permanent.
Permanence

A dynamical system is *permanent on* \( \mathbb{R}^d_{\geq 0} \) if there is some compact attracting set that does not intersect \( \partial \mathbb{R}^d_{\geq 0} \).
Permanence

A dynamical system is *permanent on* $\mathbb{R}^d_{\geq 0}$ if there is some compact attracting set that does not intersect $\partial \mathbb{R}^d_{\geq 0}$.
Polynomial Dynamical System

We are concerned with polynomial dynamical systems. We allow these to have time varying but bounded coefficients.

\[ \dot{x} = \sum_{i=1}^{n} k_i(t)x^{s_i}v_i \]

where \( x \) is our state variable and there is some \( \varepsilon > 0 \) such that \( \varepsilon < k_i(t) < \frac{1}{\varepsilon} \) for all \( t \) and \( i \).

Terminology:

- Source complexes (source vectors, sources) \( s_i \)
- Reaction vectors \( v_i \)
- Time variable kinetic coefficients \( k_i(t) \)
Euclidean embedded graph

A Euclidean embedded graph is a finite, directed, labeled graph.

- Nodes are labeled by points in $\mathbb{R}^d$. 

![Diagram of Euclidean embedded graph with labeled points]
Generating a system

To each $e_i$ edge of Euclidean embedded graph $G$, we call
- The label $s(e_i) \in \mathbb{R}^d$ of the source node the \textit{source vector}
- The label $t(e_i) \in \mathbb{R}^d$ of the target node the \textit{target vector}
- The vector $v(e_i) = t(e_i) - s(e_i)$ the \textit{reaction vector}

We can generate a polynomial dynamical system

$$\dot{x} = \sum_{e_i \in G} k_i(t) x^{s(e_i)} v(e_i)$$

where $k \in \mathbb{R}^{|E|}$ is any allowable choice of kinetic coefficients, that is $\exists \varepsilon > 0$ such that $\varepsilon < k_i(t) < \frac{1}{\varepsilon}$. 
Euclidean embedded graph & a system it generates

\[
\begin{pmatrix}
\dot{x} \\
\dot{y}
\end{pmatrix}
= k_1(t) y \begin{pmatrix}
2 \\
1
\end{pmatrix} + k_2(t) x y^2 \begin{pmatrix}
1 \\
0.5
\end{pmatrix} + k_3(t) x \begin{pmatrix}
0 \\
1
\end{pmatrix} \\
+ k_4(t) x^2 y^2 \begin{pmatrix}
-1 \\
-1
\end{pmatrix}
\]
Proof Idea

- Embed system in *differential inclusion*, $\dot{x} \in F(x)$, where $F$ is a piecewise constant set valued map.
  - Divide $\mathbb{R}^d_>$ into regions.
  - Assign to each region a cone.
- Build a nested family of forward invariant regions for this differential inclusion.
- Define a Lyapunov function using the borders of these regions as level sets.
Differential Inclusion

A differential inclusion is a dynamical system

\[ \dot{x} \in F(x) \]

where \( F \) is a set valued map. We say that a system of differential equations \( \dot{x} = f(x, t) \) is \textit{embedded} in a differential inclusion in the domain \( \Omega \) if

\[ f(x, t) \in F(x) \quad \forall x \in \Omega, t \in \mathbb{R} \]

and \textit{strictly embedded} if \( f(x, t) \in F(x)^\circ \).

We will consider differential inclusions that are piecewise constant and consist of convex cones.
\section*{\textbf{N}-cone Differential Inclusion}

A \textit{N}-cone differential inclusion, $K(N)$ is a dynamical system of the form

$$\dot{x} \in \bigcup \{N|x \in \text{fat}_\varrho(N)\} K(N)$$

where $K(N)$ is a cone for each $N \in \mathcal{N}$, for some $\varrho > 0$. 

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{diagram.png}
\end{figure}
Tropically endotactic differential inclusion

We consider the straight directions in which trajectories can \textbf{escape} to $\partial \mathbb{R}^d_\geq$ \textit{without leaving a cone of the exponential fan.} Call the collection of these directions $B^\delta(N_i)$. 

\begin{figure}
\centering
\includegraphics[width=0.5\textwidth]{tropically_endotactic_differential_inclusion.png}
\end{figure}
Tropically endotactic differential inclusion

Consider curves $C(t) \to \partial \mathbb{R}^d_{\geq 0}$. Curves $C(t) = \exp(rt)$ where $r \in \hat{N}_i$ have this property. More generally,

$$C(t) = \exp (rt + g(t)p)$$

$r \in \hat{N}, \|r\| = 1, g(t) = 1 - ae^{-\beta t}, \beta \geq 0, a \in \mathbb{R}$, and $p \in r^\perp$ are called escape curves.
Tropically endotactic differential inclusions

\[ B^\delta(N_i) \] are the tangents \( C''(t) \) to escape curves for \( t > \frac{1}{\delta} \).
Tropically endotactic differential inclusions

In two dimensions, we can find the escape directions $B^\delta(N_i)$. This fan shows all the qualitative possibilities of $B^\delta(N_i)$ in two dimensions.
Tropically endotactic differential inclusions

**Definition (Tropically endotactic differential inclusion)**

A differential inclusion is *tropically endotactic* if it can be embedded in some $\mathcal{N}$-cone differential inclusion $\mathcal{K}(\mathcal{N})$ such that for every $N \in \mathcal{N} \setminus \{1\}$, there is some $\delta > 0$ such that

$$K(N) \cap B^\delta(N)^\circ = \emptyset$$

and if $N \in \mathcal{N}$ is a face of $M \in \mathcal{N}$, then $K(M) \subseteq K(N)$.

**Theorem**

*Let $\dot{x} \in F(x)$ be a differential inclusion defined on $\mathbb{R}^2_{\geq 0}$. If $\dot{x} \in F(x)$ is tropically endotactic, then it is persistent and has bounded trajectories. Furthermore, there exists a nested family of compact regions which are forward invariant under $\dot{x} \in F(x)$, and this family covers $\mathbb{R}^2_{\geq 0}$.***
Permanence of tropically endotactic systems

We can build a compact forward invariant region, and show from this that a nested family exists.
Tropically endotactic systems

**Definition (Tropically endotactic system)**

Let $\mathcal{N}$ be a fan and $\dot{x} = f(x, t)$ a $vk$-polynomial system. The system is *tropically endotactic* if it is strictly embedded in a tropically endotactic differential inclusion.

**Definition (Tropically endotactic graph)**

Let $\mathcal{N}$ be a fan, $G$ be a Euclidean embedded graph, and $K^\delta_G(\mathcal{N})$ the associated approximation. $G$ is *tropically endotactic* any system generated by $G$ is tropically endotactic.
Permanence of tropically endotactic systems

**Theorem**

*Theorem 1* Let $\dot{x} = f(x,t)$ be a two dimensional $vk$-polynomial system and $\hat{\mathcal{N}}$ be a complete fan. If $\dot{x} = f(x,t)$ is *tropically endotactic*, then $\dot{x} = f(x,t)$ is *permanent*.
Permanence of tropically endotactic systems

We have an set of forward invariant regions from the differential inclusion that the system is strictly embedded in. This gives a Lyapunov function.
Finding the right differential inclusion

We look for regions of $\mathbb{R}^d_{\geq 0}$ in which some reaction vector “dominates”, i.e. there is $\delta > 0$ such that

$$\delta x^{s_i} > \frac{1}{\delta} x^{s_j} \quad \forall j \neq i$$

Consider $\dot{x} = f(x)$ generated by Euclidean embedded graph $G$. 
Finding the right differential inclusion: The Dominance Differential Inclusion

When this is true, the angle between $v_i$ and $\dot{x}$ decreases with $\delta$. 
Defining the Dominance Differential Inclusion

In each region, we can order the monomials according to relative magnitude: $x_1^{(2,2)} > x_1^{(1,2)} > x_1^{(1,0)} > x_1^{(0,1)}$. This is constant in each region.

(a) Source nodes  
(b) Dominance ordering fan $\hat{D}_G$.  
(c) Dominance ordering exponential fan $D_G$. 

Brunner & Craciun
UW Madison
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Defining the Dominance Differential Inclusion

Assign to each region a small solid cone $K^\delta_G(D_i)$ such that $\dot{x} \in K^\delta_G(D_i)$ when $x \in fat_a(D_i)$. Can use any other fan $\mathcal{N}$ by taking $K^\delta_G(N_i) = \bigcup\{K^\delta_G(D_i) | D_i \cap N_i \neq \emptyset\}$. 
A tropically endotactic system

Our example Euclidean embedded graph, which is not endotactic, is $\mathcal{N}$-tropically endotactic.
A tropically endotactic system

This systems is a slight modification of the well known Lotka-Volterra model of population dynamics.
References


Thank You