Math 340: Vector Spaces, Linear Dependence, Basis, Span, and not being confused

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Abstract

We are learning about vector spaces, the span of vectors, linear combinations of vectors, what a basis is for a vector space is, and how all these things go together. In this little note sheet, I'll try to focus on this last topic. It's easy to be confused by all these things, especially with how related they are and how precise we want to be with the language we use. I'll include abbreviated definitions, but the discussion will be about how to think of these topics in the context of one another.

Definitions

Before setting down the definitions (primarily from Kolman & Hill), I need the disclaimer that they won't be detailed. The idea here is that you know the definitions, and are reading this to get a better intuition about these things. Furthermore, I've put these here just as a reference. Don't read through it now. Instead, refer back whenever you aren't 100% sure about a definition that is being discussed. So skip on down to the next section. Go on. Stop reading this section. Really, it's a list of definitions. It's going to be boring.

Definition 1. A *real vector space* is a set of elements V on which we have two operations, $\oplus$, $\odot$ defined with a list of properties. Importantly, V is closed under these two operations. [1]

There are some requirements on the operations, but they should be thought of as generalizations of vector addition and scalar multiplication.

Definition 2. A *subspace* is a subset $W \subseteq V$ that is also a vector space with the operations $\oplus$ and $\odot$ from V [1].

We sometimes say W "inherits" the operations from V.

Definition 3. A *linear combination* of vectors $\boldsymbol{v}_1, ..., \boldsymbol{v}_n$ is any vector $\boldsymbol{y}$ which can be written as

$$\boldsymbol{y} = \sum_{i=1}^{n} a_i \boldsymbol{v}_i, \quad a_i \in \mathbb{R}$$

[1]

Definition 4. The *span* of vectors $\boldsymbol{v}_1, \boldsymbol{v}_2, ..., \boldsymbol{v}_n$ is the set of all linear combinations of these vectors:

$$span(\{\boldsymbol{v}_1, ..., \boldsymbol{v}_n\}) = \left\{ \boldsymbol{y} = \sum_{i=1}^{n} a_i \boldsymbol{v}_i \middle| a_i \in \mathbb{R} \right\}$$

[1]

Any set of vectors has some span. Put another way, any set of vectors spans some space.

Definition 5. A set of vectors $\boldsymbol{v}_1, ..., \boldsymbol{v}_n$ is said to be *linearly independent* if

$$\sum_{i=1}^{n} a_i \boldsymbol{v}_i = 0 \Rightarrow a_j = 0 \; j = 1, ..., n$$

[1]

In other words, the homogeneous equation has *only the trivial solution.*

Definition 6. A set of vectors $\boldsymbol{v}_1, ..., \boldsymbol{v}_n$ is said to be *linearly dependent* if there is a choice of $a_1, ..., a_n$ that are not all zero such that

$$\sum_{i=1}^{n} a_i \boldsymbol{v}_i = 0$$

[1]

Notice, we could have just defined linearly dependent as "not linearly independent", or we could have defined linearly independent as "not linearly dependent". A set of vectors is always one or the other, and never both.

Definition 7. Let V be a vector space. A *basis for* V is any set $\{\boldsymbol{v}_1, ..., \boldsymbol{v}_n\}$ such that

1. $\boldsymbol{v}_1, ..., \boldsymbol{v}_n$ span V (put another way, $span(\{\boldsymbol{v}_1, ..., \boldsymbol{v}_n\}) = V$)

2. $\boldsymbol{v}_1, ..., \boldsymbol{v}_n$ are linearly independent.

Notice that this definition requires we specify V. Saying "this set is a basis" is meaningless without saying what it is a basis for.

Definition 8. The *dimension* of a vector space V is the number of vectors in a basis for V.

Definition 9. The set of vectors $\boldsymbol{x}$ such that $A\boldsymbol{x} = \boldsymbol{0}$ is the *null space* of A.

Definition 10. The span of the columns of a matrix A is the *column space* of A.

Definition 11. The span of the rows of a matrix A is the *row space* of A.

Definition 12. The dimension of the column space of a matrix A is the *rank* of A.

Building a Vector Space: Span & Basis

A vector space is a very abstract concept. All we really need to know about a vector space to define it is that there are vectors, and that there is a (reasonable) way to add them and multiply them by scalars. We can actually go a long way without *really* specifying what any of that is, and just trust that the operations exist. But we usually want some concrete way of talking about a vector space. We want to write down examples of vectors, and we want to add them explicitly. We need some framework for doing that. We are going to call that framework a "basis".

In fact, you'll notice that I haven't started in the normal way by trying to talk about vector spaces abstractly. You do need to keep in mind that they are an abstract concept, but until you have a deeper understanding of a certain family of vector spaces it may be best not to worry about the abstract concept. The family you should start with is $\mathbb{R}^n$. That is, n-dimensional real space. In fact, I'm going to talk a lot about $\mathbb{R}^2$ and $\mathbb{R}^3$. That's for the simple reason that these are the easiest vector spaces to draw and visualize. If you can visualize $\mathbb{R}^4$ or $\mathbb{R}^{10}$ or $\mathbb{C}^3$ then you I'd like you to share your secret with me. Anyway, rather than start with an abstract definition of a vector space and then spending all our time proving such and such set with such and such operators make a vector space, I'm going to start with a few vectors and build myself a vector space. Plus, that other stuff is in your homework.

To write a vector $\boldsymbol{v} \in \mathbb{R}^3$, for example, I need a basis. You will want to write the vector

$$\boldsymbol{v} = \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix}$$

and you can. But you *have* used a basis. This is because

$$\begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = v_1 \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + v_2 \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} + v_3 \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \tag{1}$$

In other words, $\boldsymbol{v}$ is a linear combination of the vectors of the *standard basis*.

Before we can get to any fancier basis than the standard one, we need to think about what we can "build" out of any set of vectors. This is the idea of span. To build the vector $\boldsymbol{v}$, I used a linear combination of the vectors in the standard basis. I could do this because *$\boldsymbol{v}$ is in the span of those vectors.* It's fairly easy to see that any vector in $\mathbb{R}^3$ can be written as a linear combination of the vectors

$$\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right\}$$

so we can say that these vectors *span* $\mathbb{R}^3$. Don't get confused just yet. Vectors *have a span* and they *span a space.* They *span their span.* In fact they are a *spanning set* for their span, which is the space that they span. Yeah, I think that was done just to confuse you.*

It turns I didn't need to build my vector out of that particular collection of vectors. In other words, there are *other* collections of vectors that I can put together (combine) in some way involving only addition and scalar multiplication (a linear way). Yes, I can create a *linear combination* of some other vectors to get my vector. So, the question becomes can I do this with *any* set of vectors. The answer is, of course, no. First of all, if I want to write my vector as a linear combination of some other vectors, all these vectors had better be in the same vector space. There is no way to combine vectors in $\mathbb{R}^4$ and end up with a vector in $\mathbb{R}^3$ or $\mathbb{R}^5$. Next, my vector needs to be in the *span* of the others. Ok, there's that span again. Maybe it's time we talk about that.

The span of a single vector, or some nice stretching to warm us up.

Go look at the definition of span. It's definition 4. I'll wait. Did you read it? Did you understand it? Really? Ok, I guess we're done.

What's that, you might want some clarification? Oh, alright. Let's talk about span. The nice thing about $\mathbb{R}^3$ is that we can pretty easily visualize the span of a set of vectors. First, let's think about the span of a single vector. What vectors are in the span of a single vector? Well, first off, that vector is in there. If the set is $\{\boldsymbol{w}\}$, then I can write $\boldsymbol{w}$ as a linear combination of vectors in the set in the following way:

$$\boldsymbol{w} = 1\boldsymbol{w}$$

There are other vectors in there, too. For example, $2\boldsymbol{w}$, $-\boldsymbol{w}$ and $\pi\boldsymbol{w}$. In fact, any scalar multiple of $\boldsymbol{w}$ is in the span of $\{\boldsymbol{w}\}$. But, there aren't any more. It might be easy to *see* why, but it's a little hard to *say* why. That's because we haven't constructed the span of $\{\boldsymbol{w}\}$, we've only pointed out a few vectors that are in it. But we have a formula:

$$\mathit{span}(\{\boldsymbol{w}\}) = \left\{ \sum_{i=1}^{1} a_i\boldsymbol{w} \,\middle|\, a_i \in \mathbb{R} \right\} = \{\, a\boldsymbol{w} |\, a \in \mathbb{R}\} \tag{2}$$

Now we see that the span of $\{\boldsymbol{w}\}$ is all scalar multiples of $\boldsymbol{w}$, and nothing else. So what is that, geometrically? Now, we get some use out of thinking about $\mathbb{R}^3$. With a little thought, hopefully we can see that this is a line through the origin in the direction of $\boldsymbol{w}$. If

$$\boldsymbol{w} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

then we might say the span of $\{\boldsymbol{w}\}$ is the x axis.

*Another note on language: talking about "span" requires that you are talking about *two* different sets of vectors. One set (which I'll call a collection of vectors to keep them straight) is usually finite. This is the spanning set. The other is BIG. Uncountably infite in fact. This is the span of the collection. It's a vector space.

The span of two vectors, or a line of lines.

Great. The span of a single vector is a line. I can build a line out of a single vector, just by stretching and compressing it. Let's add a new vector $\boldsymbol{u}$ to the collection, and require that $\boldsymbol{u} \notin span\{\boldsymbol{w}\}$. Now, looking at the formula,

$$span(\{\boldsymbol{w}, \boldsymbol{u}\}) = \{\, a\boldsymbol{w} + b\boldsymbol{u} |\, a, b \in \mathbb{R}\} \tag{3}$$

Notice, if we take $b = 0$, *we get back the span of the single vector* $\boldsymbol{w}$. Make sure you see that. Look back at equation (2). Now look at equation (3) again. Imagine $b = 0$ in equation (3). Hopefully you are convinced that $span(\{\boldsymbol{w}\}) \subset span(\{\boldsymbol{w}, \boldsymbol{u}\})$. That is, any vector in the span of $\boldsymbol{w}$ is also in the span of $\{\boldsymbol{w}, \boldsymbol{u}\}$. Geometrically, that means that the line $span(\{\boldsymbol{w}\})$ sits inside the set $span(\{\boldsymbol{w}, \boldsymbol{u}\})$. So what does $span(\{\boldsymbol{w}, \boldsymbol{u}\})$ look like? Let's take a look at a vector $\boldsymbol{v}_1 \in span(\{\boldsymbol{w}, \boldsymbol{u}\})$. Because it's in the span of $\boldsymbol{w}$ and $\boldsymbol{u}$, we can write

$$\boldsymbol{v}_1 = a_1\boldsymbol{w} + b_1\boldsymbol{u} \tag{4}$$

Imagine we are travling around in our vector space (imagine $\mathbb{R}^3$, that's easiest). Then equation (4) gives us directions on how to get to $\boldsymbol{v}_1$ from the origin. We proceed a distance a_1 *along the line* $span(\{\boldsymbol{w}\})$ and then stop, turn in the direction of $\boldsymbol{u}$, and travel for a distance b_1. What if we had gone a distance b_1^{*}[†] in the direction of $\boldsymbol{u}$ instead? We could go anywhere on a line in the direction of $\boldsymbol{u}$ that goes through the point $a_1\boldsymbol{w}$. Now what if we change a_1 to a_1^*?? We can go anywhere on a *different* line, still in the directin of $\boldsymbol{u}$ but now through a different point. But the new point is still on the line $span(\{\boldsymbol{w}\})$. So what is $span(\{\boldsymbol{w}, \boldsymbol{u}\})$? It's the set of all linear combinations of $\boldsymbol{w}$ and $\boldsymbol{u}$. Right. That's the definition. But what does it *look like*? Well, it's a line of lines. We call that a plane.

The span of many vectors, or do I need all these?

What if I add another vector $\boldsymbol{t} \notin span(\{\boldsymbol{w}, \boldsymbol{u}\})$? Maybe you can guess. Let's start, as always, with the definition:

$$span(\{\boldsymbol{w}, \boldsymbol{u}, \boldsymbol{t}\}) = \{\, a\boldsymbol{w} + b\boldsymbol{u} + c\boldsymbol{t} |\, a, b, c \in \mathbb{R}\} \tag{5}$$

I'll start the same way I started before, by pointing out what happens when we take $c = 0$ in equation (5)[‡]. We get back $span(\{\boldsymbol{w}, \boldsymbol{u}\})$! Ok maybe that doesn't shock you. But it should give you the idea that by changing c, we get a different plane, parallel to $span(\{\boldsymbol{w}, \boldsymbol{u}\})$. Yep, by varying c we get a line of planes.

Now, if we are thinking about $\mathbb{R}^3$, we probably notice that a line of planes will fill up our whole space. If our space is bigger, like $\mathbb{R}^4$ or $\mathbb{R}^5$, we could go further. We could have a line of lines of planes, and so on. But let's stick to $\mathbb{R}^3$ for now. What happens if we try to add

[†] When we want to talk about a vector (or in this case scalar) who's defining feature is that it is similar to but not the same as some other vector $\boldsymbol{x}$ that we have already defined, we usually use $\boldsymbol{x}^*$, or $\tilde{\boldsymbol{x}}$, to remind ourselves. This is a little dangerous, because there is math that uses that notation for something more precise. We won't run in to that math, though.

[‡] In my LaTeX file (LaTeX is how mathematicians type papers) I have labeled this equation "spanwut" for the span of $\boldsymbol{w}$, $\boldsymbol{u}$, and $\boldsymbol{t}$. It looks pretty funny

another vector $\boldsymbol{s} \notin span(\{\boldsymbol{w}, \boldsymbol{u}, \boldsymbol{t}\})$? Well, we'd have a hard time, because there isn't one - we already realized *geometrically* that the span of $\{\boldsymbol{w}, \boldsymbol{u}, \boldsymbol{t}\}$ is all of $\mathbb{R}^3$. What does that mean algebraically? It means that for any vector $\boldsymbol{v} \in \mathbb{R}^3$,

$$\boldsymbol{v} = a\boldsymbol{w} + b\boldsymbol{u} + c\boldsymbol{t} \tag{6}$$

for some $a, b, c \in \mathbb{R}$. We can build any vector in $\mathbb{R}^3$ using $\boldsymbol{w}$, $\boldsymbol{u}$, and $\boldsymbol{t}$. But let's add another vector to our collection, even though it is in the span of the first three. What is the span of $\{\boldsymbol{w}, \boldsymbol{u}, \boldsymbol{t}, \boldsymbol{s}\}$?

$$span(\{\boldsymbol{w}, \boldsymbol{u}, \boldsymbol{t}, \boldsymbol{s}\}) = \{\, a\boldsymbol{w} + b\boldsymbol{u} + c\boldsymbol{t} + d\boldsymbol{s} |\, a, b, c, d \in \mathbb{R}\} \tag{7}$$

Of course, we start by noting that $span(\{\boldsymbol{w}, \boldsymbol{u}, \boldsymbol{t}\}) \subseteq span(\{\boldsymbol{w}, \boldsymbol{u}, \boldsymbol{t}, \boldsymbol{s}\})$. But, we said that $\boldsymbol{s}$ was in the span of $\{\boldsymbol{w}, \boldsymbol{u}, \boldsymbol{t}\}$. This means we can write

$$\boldsymbol{s} = a_2\boldsymbol{w} + b_2\boldsymbol{u} + c_2\boldsymbol{t} \tag{8}$$

and substitute this into equation (7):

$$\begin{aligned} span(\{\boldsymbol{w}, \boldsymbol{u}, \boldsymbol{t}, \boldsymbol{s}\}) &= \{\, a\boldsymbol{w} + b\boldsymbol{u} + c\boldsymbol{t} + d(a_2\boldsymbol{w} + b_2\boldsymbol{u} + c_2\boldsymbol{t}) |\, a, b, c, d \in \mathbb{R}\} \\ &= \{\, (a + da_2)\boldsymbol{w} + (b + db_2)\boldsymbol{u} + (c + dc_2)\boldsymbol{t} |\, a, b, c, d \in \mathbb{R}\} \end{aligned} \tag{9}$$

but that is the *the same as* the span of $\{\boldsymbol{w}, \boldsymbol{u}, \boldsymbol{t}\}$. Adding a linear combination of the vectors in our collection *did not change the span.* Once we have a set of three linearly independent vectors, we have all the building blocks we need for a three dimensional space. If our whole space is $\mathbb{R}^3$, then we have the building blocks for that space. We call this set of building blocks a *basis.*

I'm still not going to jump in to what a basis is. First, let's talk about the situation we were in with $\{\boldsymbol{w}, \boldsymbol{u}, \boldsymbol{t}, \boldsymbol{s}\}$. We had a collection of vectors, but one was a linear combination of the others. This is exactly what it means to have a collection that is *linearly dependent.* In general, we won't know which vector can be written as a linear combination of the others, so we define linearly dependent as in definition 6. But these are equivalent. In our case, we know that

$$\boldsymbol{s} = a_2\boldsymbol{w} + b_2\boldsymbol{u} + c_2\boldsymbol{t}$$

but this means that

$$\boldsymbol{s} - a_2\boldsymbol{w} + b_2\boldsymbol{u} + c_2\boldsymbol{t} = 0$$

and so the set satisfies the definition. If we knew instead that

$$a_2\boldsymbol{w} + b_2\boldsymbol{u} + c_2\boldsymbol{t} + d_2\boldsymbol{s} = 0$$

and that *one of* those coefficients was not 0 (as in definition 6), then we could similarly rearrange the equation to see that one vector could be written as a linear combination of the others. For example, if $b_2 \neq 0$, then

$$\boldsymbol{u} = -\frac{1}{b_2}(a_2\boldsymbol{w} + c_2\boldsymbol{t} + d_2\boldsymbol{s})$$

On the other hand, if

$$a\boldsymbol{w} + b\boldsymbol{u} + c\boldsymbol{t} = 0 \Rightarrow a = b = c = 0 \tag{10}$$

then we cannot manipulate the equation to write one vector as a linear combination of the others, because this would require dividing by 0. This is what we call *linearly independent.*

Simply put, a linearly dependent collection has a vector that can be written as a linear combination of the others, while a linearly independent collection does not. We can rewrite that sentence:

> A linearly dependent collection has a vector that *is in the span of the others*, while a linearly independent collection does not.

And now we see the connection between *span* and *linear dependence.* Every vector in a linearly independent collection contributes something to the span that the other vectors do not. Taking one away changes the span. On the other hand, a linearly dependent collection always contains at least one vector that adds nothing to the span of the collection - taking it away does not change the span.

Basis, or finally we talk about the important bit.

A basis *for a space* is a collection of vectors that span that space and are linearly independent. Simple enough, right? Well, there's a few things to make sure we understand here. For one, we should make a note that any vector in a basis for a space must be a vector in that space. Maybe this obvious. Recall that if a collection spans a space, then any vector in that space can be written as a linear combination of the vectors in the collection. That is, if

$$span(\boldsymbol{w}, \boldsymbol{u}, \boldsymbol{t}) = V$$

then any vector $\boldsymbol{s} \in V$ can be written

$$\boldsymbol{s} = a_2\boldsymbol{w} + b_2\boldsymbol{u} + c_2\boldsymbol{t}$$

We know that a vector space is *closed* under vector addition and scalar multiplication. This means that if $\boldsymbol{s} \in V$, that had better be true of the vectors $\boldsymbol{w}$, $\boldsymbol{u}$, & $\boldsymbol{t}$ as well.

So what is the point of having a basis for a space? Well, the first condition for a collection of vectors to be a basis of a space V is that any vector in the space can be written as a linear combination of the basis vectors (which is what we call the members of the basis) [§]. That is what it means to span V. The second condition, that the basis vectors be linearly independent[¶] means that this linear combination *is unique.* In our example, that means that there is no other choice a_2^*, b_2^*, and c_2^* such that

$$\boldsymbol{s} = a_2^*\boldsymbol{w} + b_2^*\boldsymbol{u} + c_2^*\boldsymbol{t}$$

and one or more of $a_2^* \neq a_2$, $b_2^* \neq b_2$, or $c_2^* \neq c_2$. If there was such a choice, we would know that one of $\boldsymbol{w}$, $\boldsymbol{u}$, or $\boldsymbol{t}$ could be written as a linear combination of the other two. Whoa.

[§] Another note on language: we can talk about a basis, which is a collection of vectors. We can also talk about a "basis vector", which is a *member of a basis.* Talking about a basis vector implies that there is some basis it belongs to. Calling a single vector a "basis" implies that there are no other members of that basis (which of course is possible).

[¶] from each other - the set is a linearly independent set

That's a pretty bold claim, and you probably don't see it as evident. But let's examine it by looking at what happens when we *can* write $\boldsymbol{s}$ as two different linear combinations of $\boldsymbol{w}$, $\boldsymbol{u}$, & $\boldsymbol{t}$. Assume $a_2^* \neq a_2$. Then

$$\boldsymbol{s} = a_2^*\boldsymbol{w} + b_2^*\boldsymbol{u} + c_2^*\boldsymbol{t} \quad \& \quad \boldsymbol{s} = a_2\boldsymbol{w} + b_2\boldsymbol{u} + c_2\boldsymbol{t}$$

which means

$$\mathbf{0} = \boldsymbol{s} - \boldsymbol{s} = (a_2^* - a_2)\boldsymbol{w} + (b_2^* - b_2)\boldsymbol{u} + (c_2^* - c_2)\boldsymbol{t}$$

We assumed that $a_2^* - a_2 \neq 0$, so

$$\boldsymbol{w} = \frac{b_2^* - b_2}{a_2 - a_2^*}\boldsymbol{u} + \frac{c_2^* - c_2}{a_2 - a_2^*}\boldsymbol{t}$$

and so this collection *cannot be linearly independent* - $\boldsymbol{w}$ is a linear combination of $\boldsymbol{u}$ and $\boldsymbol{t}$. Requiring that the collection *is* linearly independent prevents this from being possible.

This is the sense in which a basis is the set of essential building blocks of a space. A basis gives us everything we need to construct any vector in the vector space (as a linear combination of basis vectors) without any unnecessary blocks.

Now, we can start to see the idea of *dimension*. To choose a vector in a space, I can choose the linear combination of basis vectors that give me that vector. Any vector $\boldsymbol{s} \in V$ can be written

$$\boldsymbol{s} = a\boldsymbol{w} + b\boldsymbol{u} + c\boldsymbol{t}$$

so I can "choose" $\boldsymbol{s}$ by choosing a, b, & c. Three basis vectors, three choices, three dimensional space. Think back to that line of lines of lines from earlier. It took three *independent* directions to "fill up" $\mathbb{R}^3$. Those directions form a basis for $\mathbb{R}^3$, which is, of course, three dimensional. If I only gave myself two basis vectors and so two choices, I couldn't get every vector in my three dimensional space. If I gave myself four choices, *they could not be independent*. What do I mean by that? If I try to write

$$\boldsymbol{x} = a_3\boldsymbol{w} + b_3\boldsymbol{u} + c_3\boldsymbol{t} + d_3\boldsymbol{s}$$

I of course can. I can go ahead and choose $a = a_3, b = b_3, c = c_3$. That's fine. Now what happens when I choose d? Well if $\boldsymbol{s} = a_2\boldsymbol{w} + b_2\boldsymbol{u} + c_2\boldsymbol{t}$, I can rewrite:

$$\boldsymbol{x} = a_3\boldsymbol{w} + b_3\boldsymbol{u} + c_3\boldsymbol{t} + d_3(a_2\boldsymbol{w} + b_2\boldsymbol{u} + c_2\boldsymbol{t}) = (a_3 + d_3a_2)\boldsymbol{w} + (b_3 + d_3b_2)\boldsymbol{u} + (c_3 + d_3c_2)\boldsymbol{t}$$

and I have *changed my choice of* a, b, c.

Computing stuff, or rules 1-999 is never do a calculation if you don't know why you are doing it.

Rule 1000 is don't email me after 10pm.

Here's where we could talk all day about row reduction and checking which column has a pivot and great stuff like that. But the fact is, any calculation you have to do comes *directly from the definitions*. If we want to know if a collection is linearly independent, we solve a homogeneous equation. Well, of course we do - "linearly independent" is *defined* by a homogeneous equation. Similarly, "linear combination", "span", "null space", and "column space" are all *defined by a linear system.*

It's not a good idea to blindly carry out an algorithm. That's what computers do, and they are better at it than you will ever be. We require that you do carry out algorithms because we want you to *understand* the algorithm. We want you to see it in action so you know *why* it works. So don't do a calculation you don't understand.

The thing you need to keep in mind is one interpretation of matrix & vector multiplication. That is, if we have an $n \times m$ matrix

$$A = \begin{bmatrix} | & | & & | \\ \boldsymbol{a}_1 & \boldsymbol{a}_2 & \cdots & \boldsymbol{a}_m \\ | & | & & | \end{bmatrix}$$

we can think of the columns of A as vectors $\boldsymbol{a}_i$. Then

$$A\boldsymbol{x} = x_1 \begin{bmatrix} | \\ \boldsymbol{a}_1 \\ | \end{bmatrix} + x_2 \begin{bmatrix} | \\ \boldsymbol{a}_2 \\ | \end{bmatrix} + \cdots + x_m \begin{bmatrix} | \\ \boldsymbol{a}_m \\ | \end{bmatrix} = \sum_{i=1}^{m} x_i \boldsymbol{a}_i$$

Multiplying a vector by a matrix is the same as taking a linear combination of the columns of the matrix. So, if we have a collection of vectors $\boldsymbol{a}_i$, we can write

$$span(\{\boldsymbol{a}_i, ..., \boldsymbol{a}_m\}) = \{\boldsymbol{y} \in \mathbb{R}^n \,|\boldsymbol{y} = A\boldsymbol{x} \text{ for some } \boldsymbol{x} \in \mathbb{R}^m\}$$

Similarly, questions about linear dependence can be answered using the matrix A by rewriting the condition for linear independence as

$$A\boldsymbol{x} = 0 \Leftrightarrow \boldsymbol{x} = 0$$

References

[1] Bernard Kolman and David R. Hill. *Elementary Linear Algebra with Applications.* Pearson Education Inc., 9 edition, 2008.