Lecture 13: Legendre's equation

\[ (1-x^2) \phi'' + 2x \phi' + \lambda \phi = 0, \quad -1 < x < 1 \]

\[ s(x) = 1 - x^2, \quad p(x) = q(x) = 1 \]

Eigenfunction with different \( \lambda 's \) are orthogonal, since \( s(\pm 1) = 0 \).

So by series:

\[ \phi(x) = \sum_{n=0}^{\infty} a_n x^n \]

\[ \phi'(x) = \sum_{n=0}^{\infty} n a_n x^{n-1} \]

\[ \phi''(x) = \sum_{n=0}^{\infty} n(n-1) a_n x^{n-2} \]

Substitute in:

\( (1-x^2) \phi'' - 2x \phi' + \lambda \phi = 0 \)

Need to relabel \( \phi'' \) sum:

Let \( n' = n - 2 \)

\[ \phi''(x) = \sum_{n'=-2}^{\infty} (n'+1)(n'+2) a_{n'+2} x^{n'} \]

Now drop the prime on \( n' \), and start sum at 0 since first two terms vanish, anyways.
Combining the four, we have

$$(1-x) \phi^n - 2x \phi' + 2 \phi$$

$$= \sum_{n=0}^\infty \left[ (n+1)(n+2) a_{n+2} - n(n-1) a_n - 2n a_n + \lambda a_n \right] x^n = 0$$

Since the summation (assuming convergent) = 0, set coefficient of $x^n + 0$:

$$(n+1)(n+2) a_{n+2} - n(n-1) a_n - 2n a_n + \lambda a_n = 0$$

$$a_{n+2} = \frac{n(n-1) + 2n - \lambda}{(n+1)(n+2)} a_n$$

$$a_{n+2} = \frac{n(n+1) - \lambda}{(n+1)(n+2)} a_n$$

Recurrence relation for coefficient

Two solutions: $a_0, a_2, a_4, \ldots$ (even)

$a_1, a_3, a_5, \ldots$ (odd)

But do these converge? Ratio test:

$$R_n = \left| \frac{a_{n+2} x^{n+2}}{a_n x^n} \right| = \left| \frac{n(n+1) - \lambda}{(n+1)(n+2)} \right| |x|$$
\[ \lim_{n \to \infty} R_n = R = |x|^2. \]

So converge absolutely for \(|x| < 1\).

What about \(|x| = 1\)? More difficult.

For \(n \to 0\), note that

\[ a_{n+2} = \frac{n(n+1)-1}{(n+1)(n+2)} a_n \approx \frac{n}{n+2} a_n \]

So \(a_{n+4} \approx \frac{n+2}{n+4} a_{n+2} = \frac{n+2}{n+4} \frac{n}{n+2} a_n = \frac{n}{n+4} a_n \)

\[ a_{n+6} \approx \frac{n+4}{n+6} \frac{n}{n+4} a_n = \frac{n}{n+6} a_n \]

Easy to see: \(a_{n+2m} \approx \frac{n}{n+2m} a_n \)

So as \(m \to \infty\), \(a_{n+2m} \sim \frac{1}{2m} \) divergent.

\[ \sum a_n x^n \text{ is thus a divergent series at } |x|=1, \]

since it behaves like harmonic series \(\sum \frac{1}{n} \).

(Signs don't alternate at \(x = -1\), since only even/odd powers.)
We conclude: for each \( j \) there are two independent solutions (even, odd), but these diverge as \(|x|=1\) (one or both).

(These is tied to the fact that \((1-x^2)\phi''+...\) has vanishing coeff. for \( x(1)=1 \).)

Hence, we cannot construct regular solutions for general \( j \).

**BUT:** if \( j = m(m+1) \), \( m = 0, 1, 2, 3, ... \)
then the series terminates when \( n = m \).

\[ \Rightarrow \text{Legendre polynomials.} \]

When \( m \) is even, \( a_0 + a_2 x^2 + ... + a_m x^m \) terminates.

When \( m \) is odd, \( a_1 x + a_3 x^3 + ... + a_m x^m \) terminates.

For instance, for \( m=0 \), we have \( \phi_{2m}(x)=1 \) as a solution.

The other solution is given by the odd series:

\[ a_{n+2} = \frac{n}{n+2} a_n \quad a_1 = 1 \]

\[ a_{1+2m} = \frac{1}{1+2m} a_1 \quad \Rightarrow \quad a_{2m+1} = \frac{1}{2m+1} \]

\[ m=0: \quad \phi_{2m}(x) = 1 \quad \phi_{odd}(x) = \sum_{m=0}^{\infty} \frac{x^{2m+1}}{2m+1} \]
So, for \( n = 1, 2, 3, ... \), two solutions:

\[ P_n(x) \] are Legendre polynomials (1. functions of first kind)

\[ Q_n(x) \] are Legendre functions of the second kind

In practical problems, we usually throw out \( Q_n(x) \) since we demand regular (bounded) solution in \([-1, 1]\).

Why do some solutions of Legendre's equation diverge?

\[ (1-x^2) \phi'' - 2x \phi' + \lambda \phi = 0 \]

\[ \text{This is 0 at } x = \pm 1. \text{ Allows } \phi'' \to 0, \text{ with } (1-x^2)\phi'' \text{ finite.} \]

Let's examine the blowup at \( x = 1 \). Let \( y = 1-x \) or \( x = 1-y \). Then

\[ (2-y) y \phi'' + 2(1-y) \phi' + \lambda \phi = 0 \]

If \( \phi \) blows up as \( y \to 0^+ \), so do \( \phi' \) and \( \phi'' \).

In fact \( \phi' \) blows up faster than \( \phi \).

Let's show this. First prove:

**Lemma:** If \( f(x) \) is continuously differentiable on \([a, b]\), \( b > a \), and \( \lim_{x \to b^-} f(x) = \infty \), then \( \lim_{x \to b^-} f'(x) = \infty \).
**Proof:** Assume \( f'(x) < M, \ x \in [a, b] \).

The mean value theorem says \( f'(c) = \frac{f(x) - f(a)}{x - a}, \ a < x < b \),

for some \( c \in [a, x] \). But \( f'(c) < M \), so

\[
\frac{f(x) - f(a)}{x - a} < M \Rightarrow f(x) < f(a) + M(x - a).
\]

But this implies \( f(x) \) is bounded as \( x \to b^- \) which contradicts the assumption. \( \square \)

**Corollary:** \( \lim_{x \to b^-} \frac{f'}{f} = \infty \)

**Proof:** \( \lim_{x \to b^-} \frac{f'}{f} = \lim_{x \to b^-} (\log f)' = \lim_{x \to b^-} \frac{\log f}{f}, \ y = \log f. \)

But \( y \) has \( \lim_{x \to b^-} y = \infty \), since \( f \) itself goes to \( \infty \).

Then by the lemma \( \lim_{x \to b^-} \frac{\log f}{f} = \infty. \) \( \square \)

The Corollary says that \( f' \) goes to \( \infty \) infinitely faster than \( f \).

In the same way, \( f'' \to \infty \) faster than \( f' \).

Thus: \( |f''| > |f'| > |f|. \)
Now back to Legendre's equation in the coordinate \( y = 1 - \phi \):

\[(2 - y) y \phi'' + 2(1 - y) \phi' + \lambda \phi = 0\]

If \( \phi \to \infty \) as \( y \to 0^+ \), then so do \( \phi', \phi'' \).

The largest terms in the equation, as \( y \to 0^+ \), are

\[2y \phi'' + 2 \phi' = 0, \quad y \to 0^+\]

(At fixed \( \lambda \), there is no way \( \Delta \phi \) is as large as \( \phi' \).)

Hence:

\[y \phi'' + \phi' = 0 \implies \phi' \sim \frac{1}{y}\]

or \( \phi(y) \sim \log y \)

Hence, the series solution diverges logarithmically as \( y \to 0^+ \) (\( x \to \pm 1 \)).

This is consistent with the singularity being linked to the harmonic series, which diverges logarithmically:

\[
\log(N+1) < \sum_{n=1}^{N} \frac{1}{n} \leq 1 + \log N
\]

\[
\sum_{n=1}^{N} \frac{1}{n} \leq 1 + \int_{1}^{N+1} \frac{dx}{x-1}
\]