Homework 6

9.2.1 a) $C^{\pm} = \left(\pm\sqrt{b(r-1)},\ \pm\sqrt{b(r-1)},\ r-1\right)$

$$\dot{x} = \sigma(y-x)$$
$$\dot{y} = rx - y - xz$$
$$\dot{z} = xy - bz$$

Linearization: $A = \begin{pmatrix} -\sigma & \sigma & 0 \\ r-z & -1 & -x \\ y & x & -b \end{pmatrix}$

$$A_{\pm} = A(C_{\pm}) = \begin{pmatrix} -\sigma & \sigma & 0 \\ r-(r-1) & -1 & \mp\sqrt{b(r-1)} \\ \pm\sqrt{b(r-1)} & \pm\sqrt{b(r-1)} & -b \end{pmatrix}$$

Let $\alpha = \pm\sqrt{b(r-1)}$

$$A_{\pm} = \begin{pmatrix} -\sigma & \sigma & 0 \\ 1 & -1 & -\alpha \\ \alpha & \alpha & -b \end{pmatrix}$$

$$|A_{\pm} - \lambda I| = \begin{vmatrix} -\sigma-\lambda & \sigma & 0 \\ 1 & -1-\lambda & -\alpha \\ \alpha & \alpha & -b-\lambda \end{vmatrix} = -(\sigma+\lambda)\begin{vmatrix} -(1+\lambda) & -\alpha \\ \alpha & -(b+\lambda) \end{vmatrix} - \sigma\begin{vmatrix} 1 & -\alpha \\ \alpha & -(b+\lambda) \end{vmatrix}$$

$$= -(\sigma+\lambda)\left[(1+\lambda)(b+\lambda) + \alpha^2\right] - \sigma\left[-(b+\lambda) + \alpha^2\right]$$
$$= -(\sigma+\lambda)\left[\lambda^2 + (b+1)\lambda + \alpha^2 + \boxed{b}\right] + \sigma\left[(b+\lambda) - \alpha^2\right]$$
$$= -\lambda^3 + \lambda^2\left(-\sigma - (b+1)\right) + \lambda\left(-\sigma(b+1) - \alpha^2 + \sigma - b\right) - \sigma\alpha^2 - \sigma\alpha^2 + \cancel{b\sigma} - \cancel{b\sigma}$$
$$= 0 \qquad \alpha^2 = b(r-1)$$

$$\lambda^3 + (\sigma+b+1)\lambda^2 + \left(\sigma(b+\cancel{1}) + b(r-\cancel{1}) - \cancel{\sigma} + \cancel{b}\right)\lambda + 2\sigma b(r-1) = 0$$

$$\lambda^3 + (\sigma+b+1)\lambda^2 + b(r+\sigma)\lambda + 2\sigma b(r-1) = 0$$

b) $\lambda = i\omega \leftarrow$ imag.
$\lambda^2 = -\omega^2 \leftarrow$ real.
$\lambda^3 = -i\omega^3 \leftarrow$ imag.

Equate real / imag:

$$i\left(-\omega^3 + b(r+\sigma)\omega\right) = 0 \qquad \text{imag.}$$
$$-(\sigma+b+1)\omega^2 + 2\sigma b(r-1) = 0 \qquad \text{real}$$

Solves $\omega^2 = b(r+\sigma) = \dfrac{2\sigma b(r-1)}{\sigma+b+1}$, solve for $r=r_H$.
($\omega \neq 0$)

$$\left(b - \frac{2\sigma b}{\sigma+b+1}\right) r_H = -\sigma b - \frac{2\sigma b}{\sigma+b+1}$$

$$\cancel{b}(\sigma+b+1-2\sigma) r_H = -\sigma\cancel{b}(\sigma+b+1+2)$$

$$r_H = \sigma\left(\frac{\sigma+b+3}{\sigma-b-1}\right)$$ Need $r_H > 0$, so $\sigma > b+1$.
(otherwise $C_\pm$ stable)

c) $\lambda^3 + (\sigma+b+1)\lambda^2 + b(r+\sigma)\lambda + 2\sigma b(r-1) = (\lambda+i\omega)(\lambda-i\omega)(\lambda-c)$

third eigenval.

$$= \lambda^3 + (\cdot\cdot)\lambda^2 + (\cdot\cdot)\lambda - c\omega^2$$

Hence, $c\omega^2 = -2\sigma b(r-1)$

$$c = -\frac{2\sigma b(r-1)}{\omega^2} = -2\sigma b\frac{(\sigma+b+1)}{2\sigma b} = -(\sigma+b+1) < 0,$$ as required.

Another argument: $\dfrac{\partial\dot x}{\partial x} + \dfrac{\partial \dot y}{\partial y} + \dfrac{\partial \dot z}{\partial z} = -(\sigma+b+1) = (\text{sum of eigenvalues})$
$= i\omega - i\omega + c$

9.2.2 Let $V = rx^2 + \sigma y^2 + \sigma(z-2r)^2$

$$\dot V = 2rx\dot x + 2\sigma y\dot y + 2\sigma(z-2r)\dot z$$
$$= 2r\sigma x(y-x) + 2\sigma y(rx - y - xz) + 2\sigma(z-2r)(xy-bz)$$
$$= \cancel{4r\sigma xy} - 2r\sigma x^2 - 2\sigma y^2 - \cancel{2\sigma xyz} + 2\sigma(\cancel{xyz} - \cancel{2rxy} - bz^2 + 2brz)$$
$$= -2r\sigma x^2 - 2\sigma y^2 - 2\sigma bz(z-2r) + 4\sigma brz$$

Now assume we are on the boundary of E, so $rx^2 + \sigma y^2 + \sigma(z-2r)^2 = V = \text{const.}$
$= C$
Need $\dot V < 0$ there, so that V must decrease
$(V>0)$

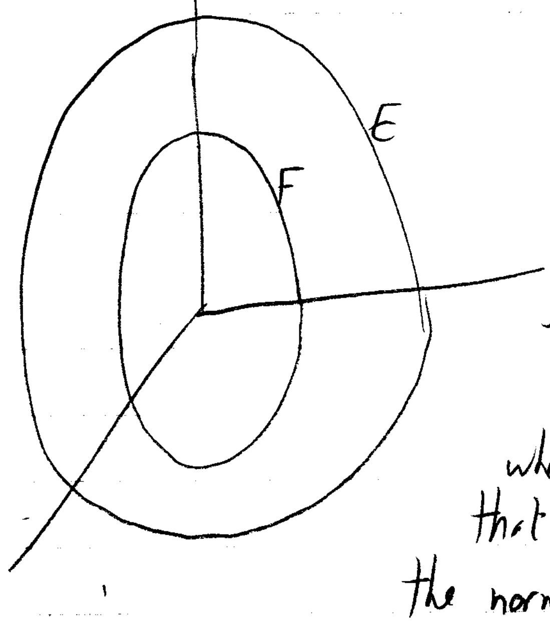
$$-2r\sigma x^2 - 2\sigma y^2 - 2\sigma b(z^2 - 2rz) < 0$$

$$rx^2 + y^2 + b\left[(z-r)^2 - r^2\right] > 0$$

$$rx^2 + y^2 + b(z-r)^2 > br^2$$

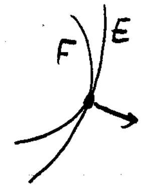
$W(x,y,z) = 1$

Let F be the ellipsoid $\dfrac{x^2}{br} + \dfrac{y^2}{br^2} + \dfrac{(z-r)^2}{r^2} = 1$



Choose C large enough so that F fits inside E. Then $\dot{V} < 0$ on E, so trajectories can't leave.

"stiffer challenge":
The optimal value of C occurs when the ellipsoids touch, which means that $x, y, z \in E$ and F, but also the normals are aligned.

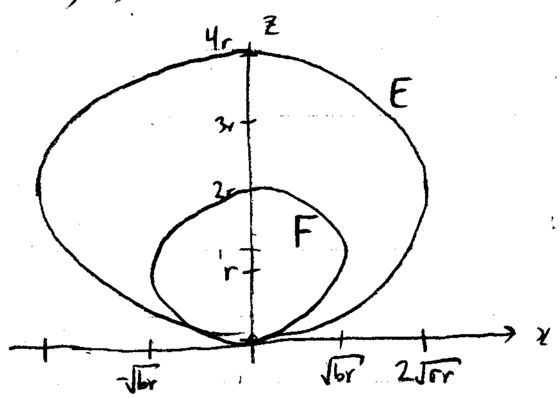


System of 5 equations: $V(x,y,z) = C$

$W(x,y,z) = 1$

$\nabla V = k \nabla W$ ← normals are aligned, 3 equations

Solve for x, y, z, k, C. Find: $x = y = z = 0$, $k = \dfrac{2\sigma}{b}$, $\boxed{C = 4\sigma r^2}$ *



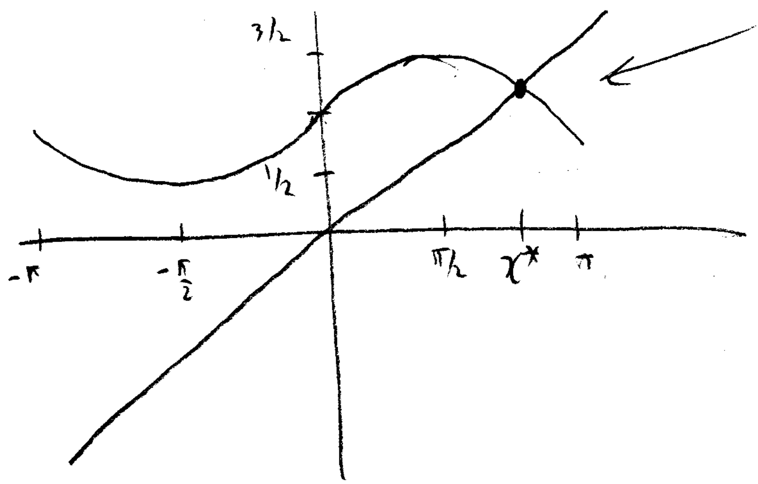
E: $\dfrac{x^2}{4\sigma r} + \dfrac{y^2}{4r^2} + \dfrac{(z-2r)^2}{4r^2} = 1$

F: $\dfrac{x^2}{br} + \dfrac{y^2}{br^2} + \dfrac{(z-r)^2}{r^2} = 1$

* There are a few cases to consider. This works for $b < \min(2, 2\sigma)$

10.1.10 $x_{n+1} = 1 + \frac{1}{2}\sin x_n$ Fixed pt: $x^* = 1 + \frac{1}{2}\sin x^*$

Pictorially:



Stability:

$|f'(x^*)| = \left|\frac{1}{2}\cos x^*\right| < 1$

So stable.

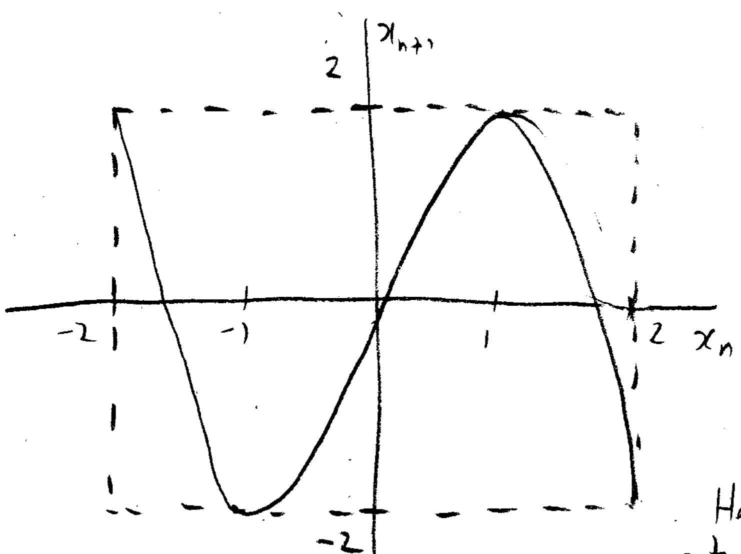
10.1.11 $x_{n+1} = 3x_n - x_n^3$

a) $x^* = 3x^* - x^{*3} \Rightarrow x^{*3} - 2x^* = 0 \Rightarrow x^*(x^{*2} - 2) = 0$

$x^* = 0, \pm\sqrt{2}$

Stability: $f'(x^*) = 3(1 - x^{*2}) = \begin{cases} 3, & x^* = 0 \leftarrow \text{stable} \\ -3, & x^* = \pm\sqrt{2} \leftarrow \text{unstable} \end{cases}$

d)



The image of $[-2,2]$ is $[-2,2]$. So any point that starts in $[-2,2)$ must stay there. So these orbits remain bounded.

$\left|\frac{f(x)}{x}\right| = |3 - x^2| > 1$ for $|x| > 2$.

Hence, initial conditions that start outside the interval must grow without bound, so $|x_n| \to \infty$

10.3.1 $x_{n+1} = f(x_n) = r x_n (1 - x_n)$ ① $f'(x) = r - 2rx = 0$ for superstable
② $f(x) = x = rx(1-x)$ for fixed point

So $x = 1/2$ from ①, and ② $r\left(\cancel{\tfrac{1}{2}}\right)\left(1 - \tfrac{1}{2}\right) = \cancel{\tfrac{1}{2}}$

$$\boxed{r = 2}$$

10.3.4 $x_{n+1} = f(x_n) = x_n^2 + c$

a) $x^* = (x^*)^2 + c \Rightarrow x^* = \dfrac{1 \pm \sqrt{1-4c}}{2}$

$c < \frac{1}{4}$ two fixed pts
$c > \frac{1}{4}$ no fixed pts

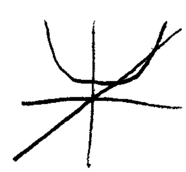
Stability: $f'(x^*) = 2x^* = 1 \pm \sqrt{1-4c}$

The "+" solution is always unstable for $c < \frac{1}{4}$.

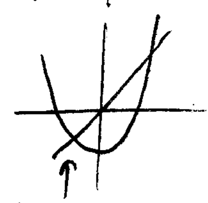
The "−" solution is unstable if $1 - \sqrt{1-4c} < -1$

$$\begin{aligned} 2 &< \sqrt{1-4c} \\ 4 &< 1 - 4c \\ 3 &< -4c \\ -3 &> 4c \\ c &< -3/4 \end{aligned}$$

There is a tangent bifurcation at $c = \frac{1}{4}$



There is a flip bifurcation at $c = \frac{-3}{4}$



slope near −1

c) $f(f(x)) = (x^2+c)^2 + c = x^4 + 2cx^2 + c^2 + c = x$ period-2

Let $p(x) = x^4 + 2cx^2 - x + c(c+1) = 0$ for period-2.

Fixed point $x^2 - x + c = 0$ is also a solution, so eliminate:

$$\begin{array}{l|l} x^4 + 2cx^2 - x + c(c+1) & x^2 - x + c \\ \hline \underline{x^4 - x^3 + cx^2} & x^2 + x + (c+1) \\ \quad x^3 + cx^2 - x & \\ - \quad \underline{x^3 - x^2 + cx} & \\ \quad (c+1)x^2 - (c+1)x + (c+1)c & \end{array}$$

Hence, $p(x) = \underbrace{(x^2 + x + (c+1))}_{=0 \text{ for period-2 orbit}} \overbrace{(x^2 - x + c)}^{\text{fixed pts}}$

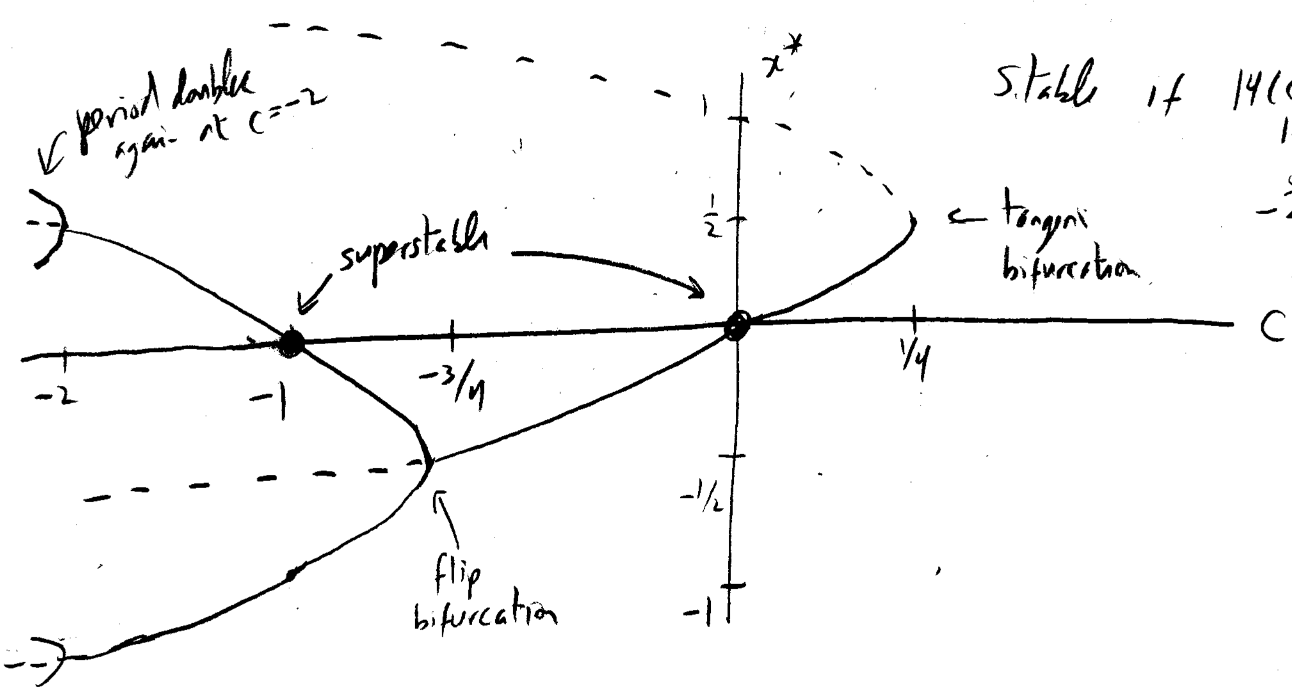
$\Rightarrow x = \dfrac{-1 \pm \sqrt{1-4(c+1)}}{2}$

$a, b = \dfrac{-1 \pm \sqrt{-3-4c}}{2}$

Stability: $\lambda = f'(a) f'(b) = (2a)(2b) = 4ab$
$= 4(c+1)$

Superstable if $\lambda = 0$
$\Rightarrow c = -1$.

Stable if $|4(c+1)| < 1$
$|c+1| < 1/4$
$-\frac{5}{4} < c < -3/4$



10.3.5. $x_{n+1} = r x_n (1 - x_n)$.

$x_n = a y_n + b$, $\quad x_{n+1} = a y_{n+1} + b$

$a y_{n+1} + b = r(a y_n + b)(1 - a y_n - b)$

$a y_{n+1} = r(-a^2 y_n^2 + y_n(-ab + a(1-b)) + b(1-b)) - b$

$y_{n+1} = -ra y_n^2 + r(1-2b) y_n + \frac{rb}{a}(1-b) - \frac{b}{a} = y_n^2 + c$

So let $b = \frac{1}{2}$, $\quad a = -\frac{1}{r}$.

$c = \frac{rb}{a}(1-b) - \frac{b}{a} = \frac{b}{a}[r(1-b) - 1] = -\frac{r}{2}\left(\frac{r}{2} - 1\right) = \frac{r}{2}\left(1 - \frac{r}{2}\right)$

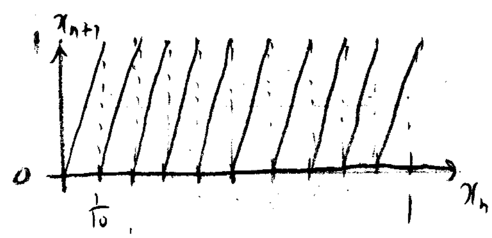
Note that there is a maximum value of c given r:

$c'(r) = \frac{d}{dr} \frac{1}{2}\left(r - \frac{1}{2} r^2\right) = \frac{1}{2} - \frac{1}{2} r = 0$ for $r = 1$.

$c(1) = \frac{1}{2}\frac{1}{2} = \frac{1}{4}$

Hence, there is no value of r that can give $c > 1/4$.

10.3.7. $x_{n+1} = 10 x_n \bmod 1$. a)



b) $x_n \in [0, 1]$, so $x_n = 0.a_0 a_1 a_2 a_3 \ldots$ $\quad a_i = 0, 1, 2, \cdots, 8, 9$.

↑ decimal representation.

$x_{n+1} = 10 x_n \bmod 1 = (a_0 . a_1 a_2 a_3 \ldots) \bmod 1 = 0.a_1 a_2 a_3 a_4 \ldots$ ← shifted by 1 to the left

So fixed points are of the form $x^* = 0.a_0 a_0 a_0 \ldots$

Note that $0.9999\ldots = 1$, so there are 9 fixed points

$\frac{1}{9} = 0.\overset{0}{1}111 \ldots$ etc. $\quad x^* = \frac{m}{9}$, $\quad n = 0, 1, 2, \ldots, 8$.

c) Can make a periodic point of period m by choosing

$$x = 0.\underbrace{a_0 a_1 a_2 \cdots a_{p-1}}_{\text{block of } p \text{ digits}} \underbrace{a_0 a_1 a_2 \cdots a_{m-1}}_{\text{same block}} a_0 a_1 a_2 \cdots a_{p-1} a_0 \cdots$$

as long as each block does not contain shorter repeated subsequences

$\Rightarrow$ all rational numbers are periodic points

d) Non-repeating sequences $\Rightarrow$ all irrational numbers.

e) $f'(x) = 10$, for all x, so all orbits unstable.

Lyapunov exponent $\lambda = \lim_{n \to \infty} \frac{1}{n} \sum_{k=0}^{n-1} \ln |f'(x_k)| = \ln 10 > 0$.

so sensitive dependence.