QUESTION 1

(____ / 20)

On the homework, you studied the equation

$$m\dot{v} = mg - kv^2$$

for a skydiver with downward speed $v(t) \geq 0$, subject to a frictional force kv^2. However, this equation predicts nonsense for $v < 0$: in particular, it gives a fixed point $v^* = -V$, where $V = \sqrt{mg/k}$ is the terminal velocity. It also predicts that a skydiver with initial velocity $v_0 < -V$ (i.e., moving upward faster than V) should keep *accelerating* upward!

To remedy this, we modify the equation to

$$m\dot{v} = mg - k|v|\,v \tag{1}$$

where $|v|$ is the absolute value of v. This equation is now valid for v positive or negative.

(a) Introduce the rescaled variables $\nu = v/a$, $\tau = t/T$. Find the value of a and T that transform equation (1) to

$$\frac{d\nu}{d\tau} = 1 - |\nu|\,\nu \tag{2}$$

(b) Draw the phase portrait for equation (2), indicating the stability of fixed points, if any.

(c) Explain why the new equation (2) fixes the problems described above.

$\nu(\tau) \rightarrow$ (d) Solve for $\nu(t)$, with initial condition $\nu(0) = -1$.

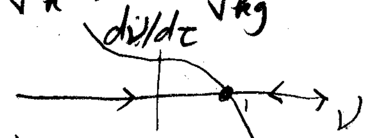
(e) Do you expect solutions to become non-unique when $\nu = 0$? Justify your answer mathematically.

ANSWER

3 (a) $\nu = v/a$, $\tau = t/T$: $\frac{ma}{T}\frac{d\nu}{d\tau} = mg - ka^2|\nu|\nu \Leftrightarrow \frac{d\nu}{d\tau} = \frac{Tg}{a} - \frac{kaT}{m}|\nu|\nu$

$T = \frac{a}{g}$, $\frac{kaT}{m} = \frac{ka^2}{mg} = 1 \Leftrightarrow a = \sqrt{\frac{mg}{k}}$, $T = \sqrt{\frac{m}{kg}}$

4 (b) Fixed points: $|\nu|\nu = 1 \Rightarrow \nu = 1$



3 (c) There is no longer a fixed point at $\nu = -1$.

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(d) $\nu(0) = -1$: $\quad \dfrac{d\nu}{d\tau} = 1 - (-\nu)\nu = 1 + \nu^2$, since $\nu < 0$.

$$\int_{-1}^{\nu} \frac{d\nu}{1+\nu^2} = \tau \iff \arctan \nu \Big|_{-1}^{\nu} = \tau$$

$\arctan(-1) = -\frac{\pi}{4}$, since $\dfrac{\sin(-\pi/4)}{\cos(-\pi/4)} = \dfrac{-\sqrt{2}/2}{\sqrt{2}/2} = -1$

$$\arctan \nu + \frac{\pi}{4} = \tau$$

$$\arctan \nu = \tau - \frac{\pi}{4}$$

5 $\nu = \tan\left(\tau - \frac{\pi}{4}\right)$ But this is valid only for $\nu \leq 0$, that is for $0 \leq \tau \leq \pi/4$

For $\tau > \pi/4$, we have $\nu > 0$, so $\dfrac{d\nu}{d\tau} = 1 - \nu^2$, with $\nu(\frac{\pi}{4}) = 0$ as initial condition (continuous)

3 $$\int_0^{\nu} \frac{d\nu}{1-\nu^2} = \int_{\pi/4}^{\tau} d\tau \iff \operatorname{arctanh} \nu = \tau - \frac{\pi}{4}$$

$$\nu = \tanh\left(\tau - \frac{\pi}{4}\right), \quad \tau > \frac{\pi}{4}.$$

$$\therefore \quad \nu(\tau) = \begin{cases} \tan\left(\tau - \frac{\pi}{4}\right), & 0 \leq \tau \leq \frac{\pi}{4} \quad \leftarrow \text{particle is moving up } (\nu<0) \\ \tanh\left(\tau - \frac{\pi}{4}\right), & \tau > \frac{\pi}{4} \quad \leftarrow \text{particle is moving down } (\nu>0) \end{cases}$$

(e) Uniqueness requires continuity of $f(\nu) = 1 - |\nu|\nu$ and $f'(\nu)$ at $\nu = 0$.

2 $\lim_{\nu \to 0^+} f(\nu) = \lim_{\nu \to 0^-} f(\nu) = 1$, so $f(\nu)$ continuous at 0.

$f'(\nu) = \begin{cases} 2\nu, & \nu < 0 \\ -2\nu, & \nu > 0 \end{cases}$, so $\lim_{\nu \to 0^+} f'(\nu) = \lim_{\nu \to 0^-} f'(\nu) = 0$, so $f'(\nu)$ continuous at $\nu = 0$.

Hence, solutions are unique.

QUESTION 2 (____/20)

On the homework, you studied the equation

$$\frac{\dot N}{N} = r - a(N-b)^2$$

for the population $N(t)$ of a species. (All the constants are positive.) This incorporates the 'Allee effect,' where the growth rate $\dot N/N$ is largest for some intermediate value of $N(t)$.

(a) Introduce the rescaled variables $n = N/\alpha$, $\tau = t/T$. Find the values of α, T, and ρ that transform the above equation into

$$\frac{1}{n}\frac{dn}{d\tau} \rightarrow \frac{\dot n}{n} = \rho - (n-1)^2$$

(b) Draw the phase portraits for $\rho > 1$ and $\rho < 1$. What sort of bifurcation occurs at $\rho = 1$?

(c) For $\rho > 1$, the system behaves like the logistic growth equation. Physically, what is the new type of behavior that can occur for $\rho < 1$?

ANSWER

4

(a) $n = \frac{N}{\alpha}$, $\tau = \frac{t}{T}$: $\quad \frac{1}{T}\frac{1}{n}\frac{dn}{d\tau} = r - a(\alpha n - b)^2$

$$\frac{1}{n}\frac{dn}{d\tau} = rT - \alpha^2 Ta\left(n - \frac{b}{\alpha}\right)^2$$

Set $\rho = rT$, $\alpha = b$, $T = \frac{1}{a\alpha^2} = \frac{1}{ab^2} \Rightarrow \rho = \frac{r}{ab^2}$

$$\therefore \frac{1}{n}\frac{dn}{d\tau} = \rho - (n-1)^2$$

(b) $\frac{dn}{d\tau} = n(\rho - (n-1)^2)$. Fixed points at $n = 0$, $(n-1)^2 = \rho$, $n = 1 \pm \sqrt{\rho}$.

$\frac{dn}{d\tau} = (\rho - 1)n + O(n^2)$, so $n = 0$ stable for $\rho < 1$, unstable for $\rho > 1$.

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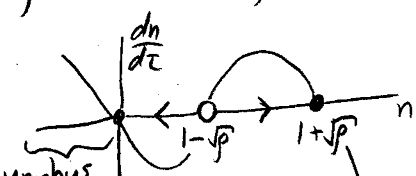
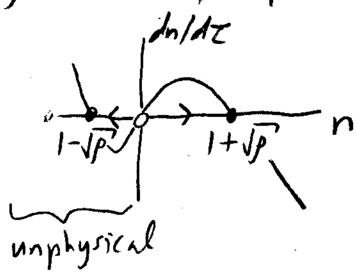
$\rho < 1$:



$\rho > 1$:



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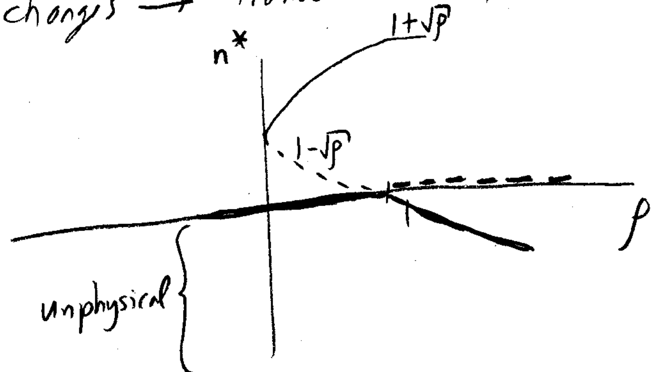
(b) (continued) The number of fixed points is the
same before/after bifurcation, but stability
4 changes $\rightarrow$ transcritical bifurcation at $\rho = 1$



4 (c) An initial condition with $n < 1-\sqrt{\rho}$ leads to extinction,
for $\rho < 1$.

QUESTION 3 (____/ 20)

For the equation

$$\dot{x} = rx - \frac{x^2}{1+x^2}$$

(a) Find all the fixed points, and find the values of r where the number of fixed points changes.
(b) Using linear stability analysis, find the values of r for which the fixed point $x^* = 0$ is stable.
(c) Draw the bifurcation diagram, indicating the values of r where bifurcations occur, and the type of bifurcation. [Hint: for the fixed points with $x \neq 0$, it might be easier to plot r as a function of x^*, rather than the other way around.]

ANSWER

4 (a) $x^* = 0$ and $r(1+x^{*2}) - x^* = 0$

$x^{*2} - \frac{1}{r}x^* + 1 = 0 \Rightarrow x^* = \dfrac{\frac{1}{r} \pm \sqrt{r^{-2}-4}}{2} = \dfrac{1 \pm \sqrt{1-4r^2}}{2r}$

Need $1-4r^2 > 0 \Leftrightarrow \frac{1}{4} > r^2 \Leftrightarrow \frac{1}{2} > |r|$ for 3 fixed pts.

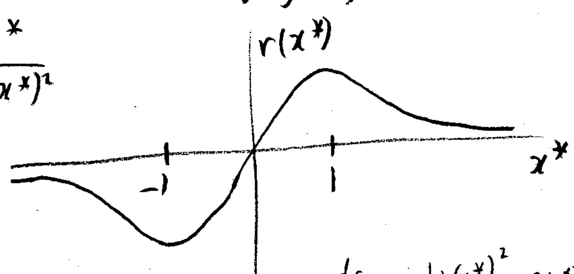
2 1 fixed point for $|r| > \frac{1}{2}$, 3 fixed points for $|r| < \frac{1}{2}$, 2 f.p. for $|r| = \frac{1}{2}$.

# of fixed points changes at $r_c = \pm\frac{1}{2}$.

4 (b) $\dot{x} = rx + O(x^2)$, so $x^* = 0$ is stable for $r<0$, unstable for $r>0$.

(or take derivative of $f(x)$, evaluate at $x=0$ to find $f'(0) = r$.)

(c) $r = \dfrac{x^*}{1+(x^*)^2}$



← use this to plot bifurcation diagram (over →)

$\dfrac{dr}{dx^*} = \dfrac{1+(x^*)^2 - 2(x^*)^2}{(1+(x^*)^2)^2} = \dfrac{1-(x^*)^2}{(1+(x^*)^2)^2} = 0$ at $x^* = \pm 1$

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(c) (continued)

2 • saddle-node bifurcations at $r_c = \pm 1/2$

2 • transcritical bifurcation at $r = 0$.

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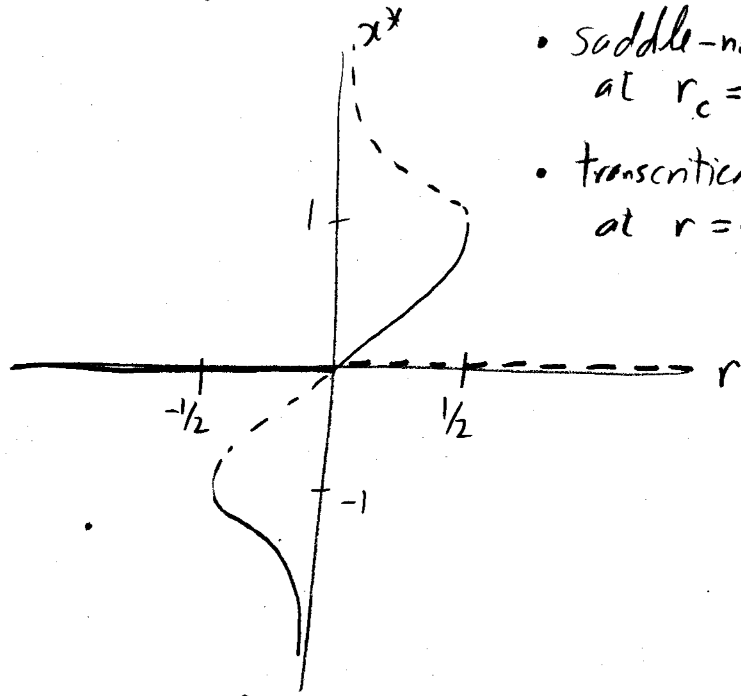

stability must alternate, and we know stability of $x^* = 0$.