Braids

23/1/2008

Lecture 1

Books: (on reserve in library)

"Braids, Links, & Mapping Class Groups"
by Joan Birman (PUP 1975)

"Braids & Coverings" by Van Lundsgaard Hansen
(LMS 1989)

For the first few parts of his book VLH follows Birman closely, but his perspective is often different enough to be useful.

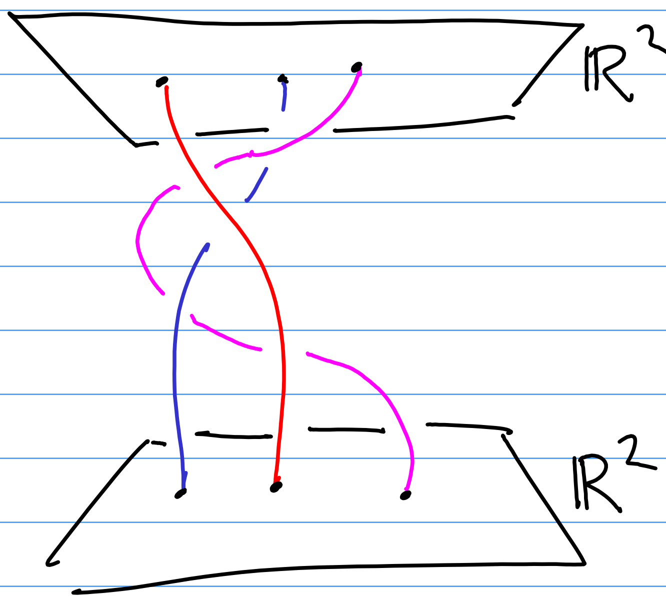
More importantly, we will consult many research papers as we go. I will try to post references on my website. I hope to get students to make short presentations on specific topics / proofs.

Often the proofs presented will be "sketches". It is more important to understand than to fill in all the blanks!

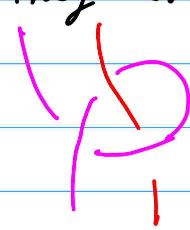
What are braids?

Braids are of course closely related to knots, but in many ways are simpler. They have a natural abstract algebraic description, and they generalize more readily and usefully to other spaces than $\mathbb{R}^3$. The physical applications of braids are less common and obvious than knots, but they are important just the same.

The man behind braids is Emil Artin (1898-1962). The basic setting is simple: connect strings between fixed points, or punctures, in $\mathbb{R}^2$

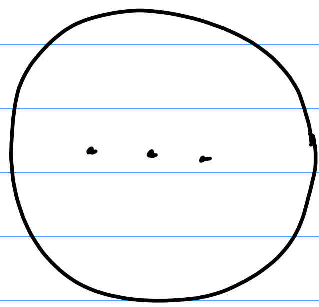


- The strings can never cross
- They never "go back" on themselves.

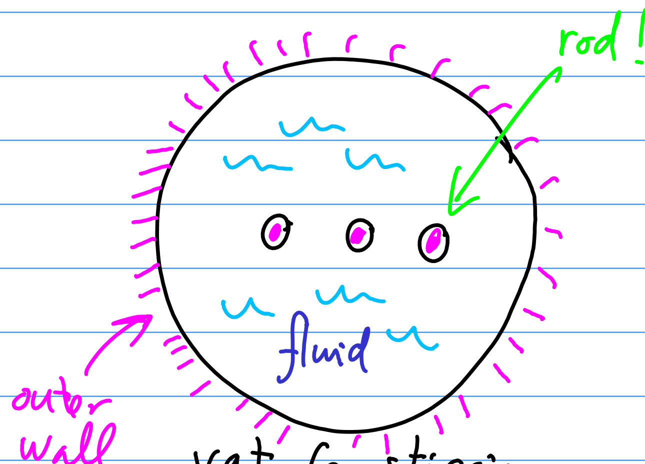


The rule is that two braids are equivalent if their strings can be deformed into each other w/o crossing.

The second constraint (that the strings never go back) suggests a particular use for braids which is close to my hearts: consider a disk with some holes in it:



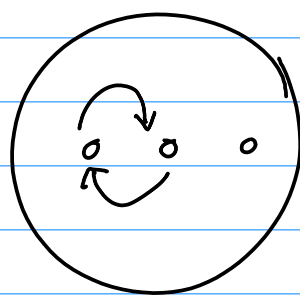
disc



vat for stirring fluid.

A punctured disk is topologically equivalent to a two-dimensional stirring device! Now to

Now let's say we move the rods around as follows:



We interchange the 1st & 2nd rod.

Let $z(t) = (z_1(t), z_2(t), z_3(t))$, where each $z_j(t) = x_j(t) + iy_j(t)$ is the position of rod j. $[D^2 = \text{disk} \in \mathbb{C}]$

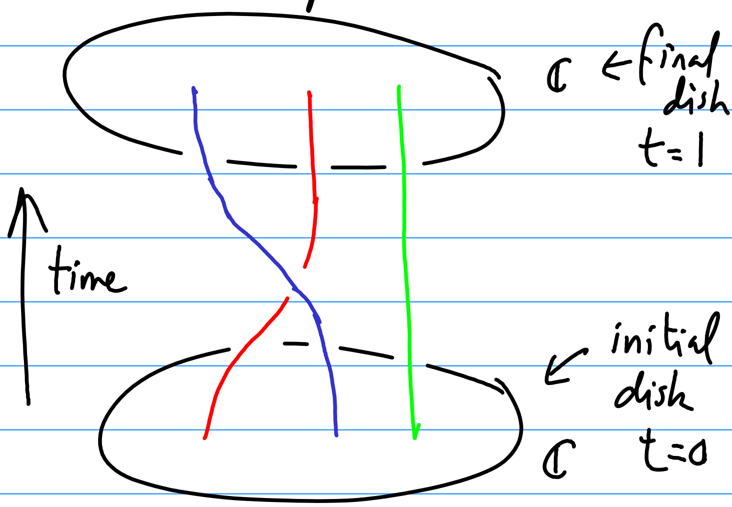
So $z_j(t) \in \mathbb{C}$, so $z(t) \in \mathbb{C} \times \mathbb{C} \times \mathbb{C}$.

We have $|z_j(t)|^2 \leq 1$ (disk of radius 1), and

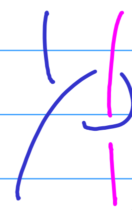
$$z_j(t) \neq z_k(t) \quad \text{for } j \neq k.$$

If we consider the motion of our rods as taking place during the interval $t \in [0,1]$, the we can plot the rod motions as follows.

up, but this is a convention!



This looks a lot like a braid! In particular, the fact that we never see

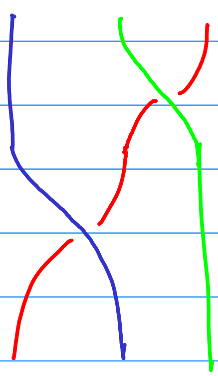


is interpreted as time never flowing backwards!

The motion of rods (or, more generally, periodic orbits) thus maps naturally to braids, and vice-versa.

Consider now a sequence of two rod motions.

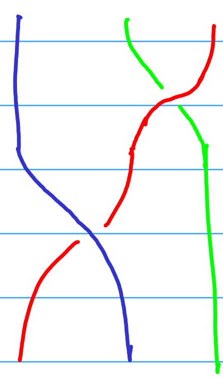
1

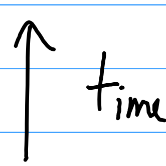


vs

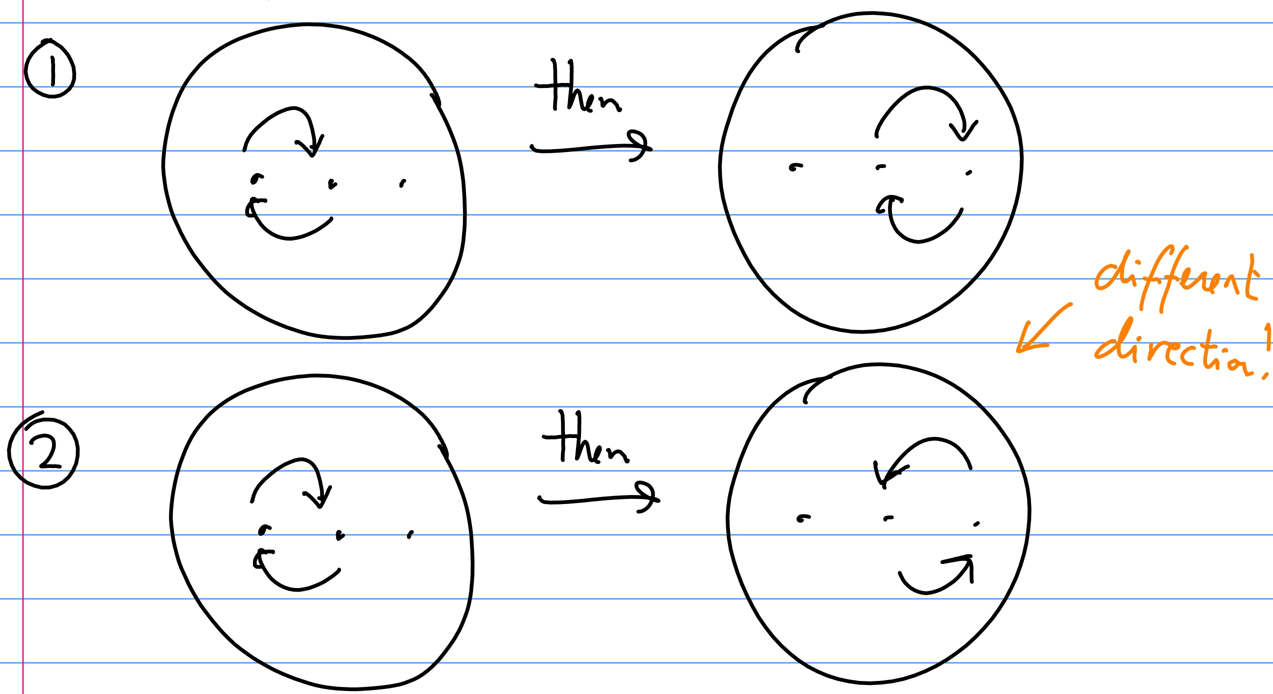
Note the subtle difference!

2

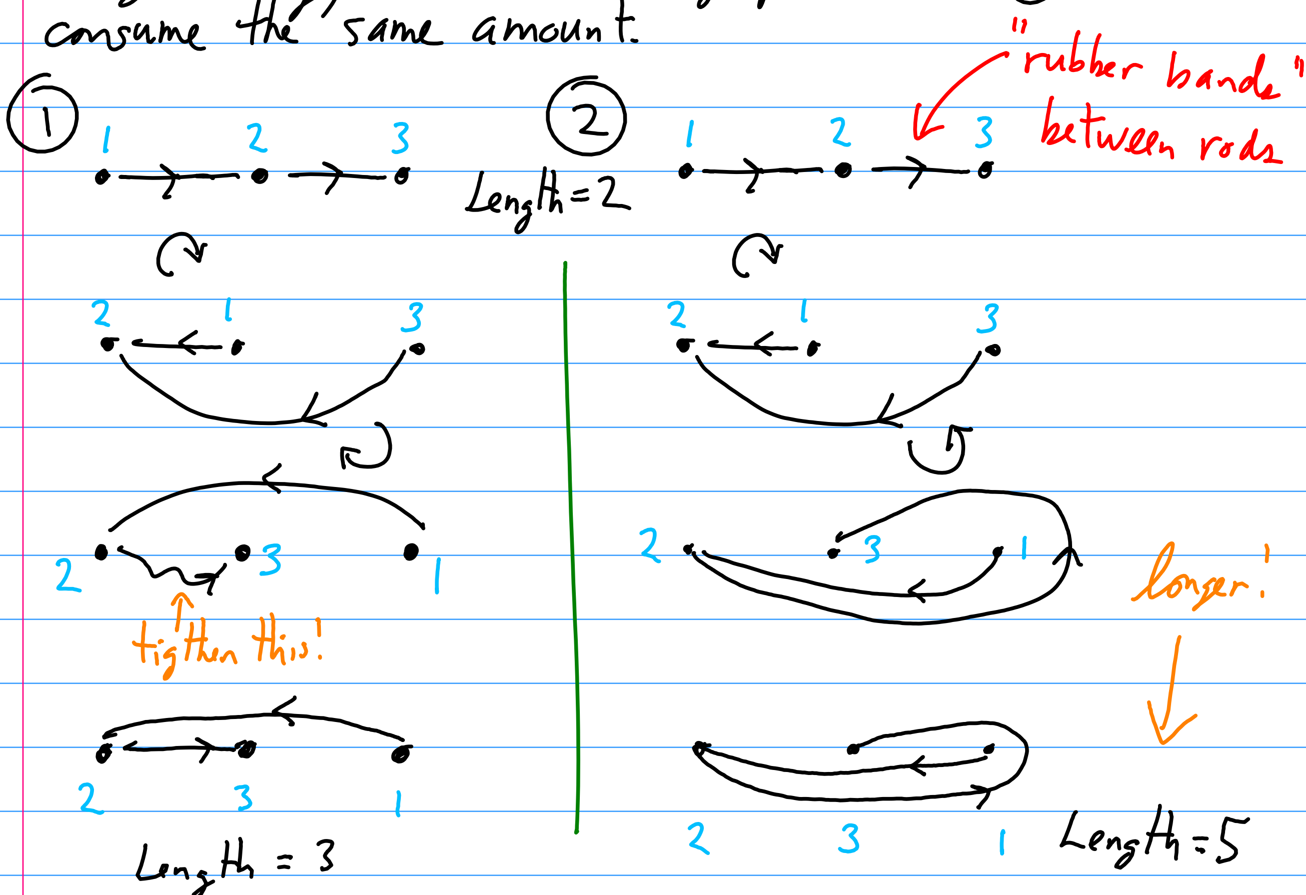




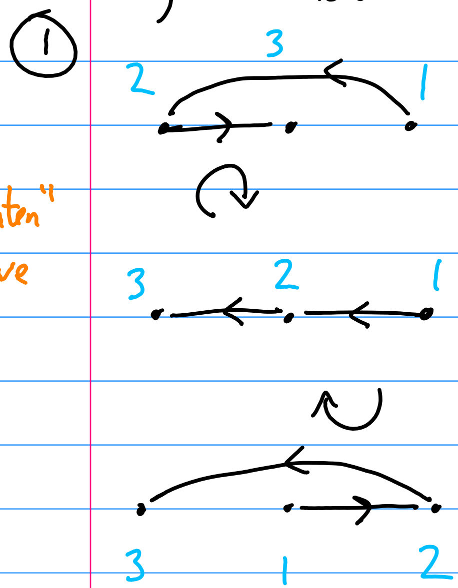
Physically, these correspond to:



How could reversing the direction of the second switch possibly matter? Well, it makes a world of difference! Energetically, the "stirring protocols" ① & ② consume the same amount.

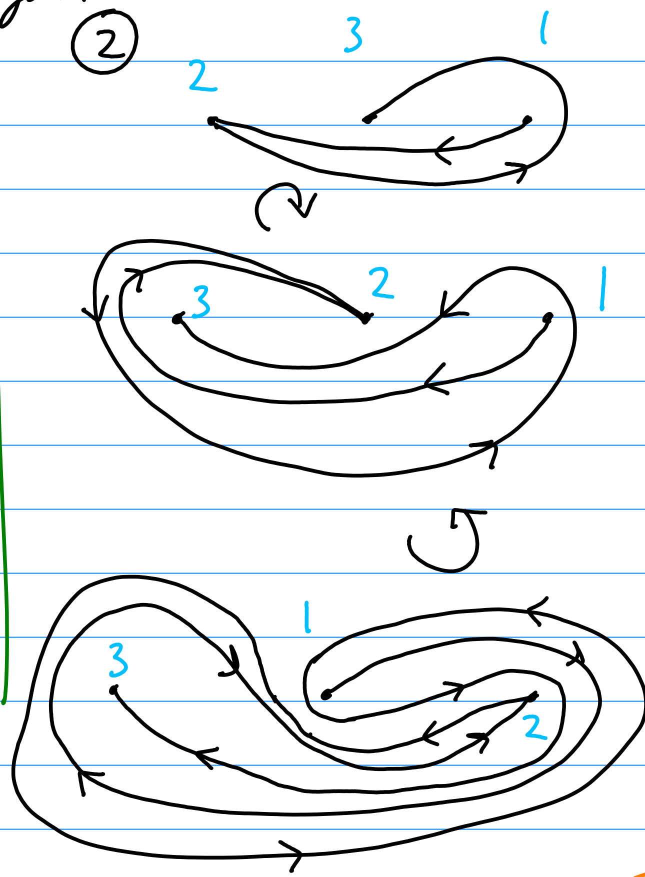


Ok, but now do it again:



We "tighten" as we go

Length = 3



After "pulling tight", of course $\longrightarrow$ Length $= 5+8 = 13$

Check: $\begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 5 & 3 \\ 3 & 1 \end{pmatrix}\begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 8 \\ 5 \end{pmatrix}.$

We will see where these come from, later!

$= 13$

If we keep applying protocol 2, the length of the rubber bands will grow asymptotically as

$$\lambda = \left(\frac{1+\sqrt{5}}{2}\right)^2 = \frac{3+\sqrt{5}}{2} \simeq 2.6180$$

Golden ratio!

Compare to $\sqrt{\frac{13}{2}} \simeq 2.55$ Close!

To sum up: In case 1 the length of the line grew very little (factor of 3/2); in case 2 it grew by a factor of 13/2! Clearly, something better is happening in case 2. During the semester, we will discover what that is.

In a nutshell, the braid traced out by the rods labels an isotopy class of homeomorphisms of the punctured disk. In case 1, the isotopy class is called finite-order. In case 2, it is pseudo-Anosov. (pA)

pA homeomorphisms have positive topological entropy, and as a consequence curves like our rubber band grow exponentially under application of the homeo.

All these concepts will be covered in more details in subsequent lectures.