25/1/2008

Braids – Lecture 2: Definition of Braids

Rolfsen (see link from class website) defines braids in 6 different ways. We will look at a few of them here.

1. Braids as particle dances: This is the definition we alluded to in the first lecture. Consider n particles located at points in the Euclidean plane $\mathbb{E}^2$, which we regard here as points in the complex plane $\mathbb{C}$. (We often call these points "punctures".) Denote the points by a vector $z = (z_1, \dots, z_n)$. Now assume the points can move,

$$z(t) = (z_1(t), \dots, z_n(t)) \qquad z_j(t) \in \mathbb{C}$$

$$t \in [0,1]$$

such that:

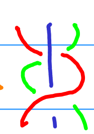
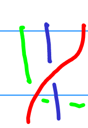
$$z_j(t) \neq z_k(t), \quad j \neq k$$

and

$$\{z_1(0), \dots, z_n(0)\} = \{z_1(1), \dots, z_n(1)\} \text{ setwise } (*)$$

The specification "setwise" is important since it means that the particles can be permuted amongst themselves.

If we replace $(*)$ by the stronger condition $z_j(0) = z_j(t)$, the strings are always anchored at the same position. we get a pure braid. In a pure braid, the strings are always anchored at the same position.

pure →     ← impure

The "particle dance" viewpoint is of course the best for describing periodic orbits or stirring rods.

2. Braids as strings in 3D: Simply embed $\mathbb{E}^2 \times [0,1]$ in $\mathbb{E}^3$. This is the best way to visualize braids. Also known as *geometric* or *physical* braids.

We shall leave Rolfsen's 4 other definitions until later. For now we use 1 & 2 and treat them as equivalent.

Equivalence of braids: Intuitively, we should be able to "deform" a braid without really changing it, as long as its strands do not cross.

Let β & β' be geometric braids, with the same base points which are both subsets of $\mathbb{E}^2 \times I$. Then we write $\beta \approx \beta'$ if there is an isotopic deformation $F_s : \mathbb{E}^2 \times I \to \mathbb{E}^2 \times I$ which is the identity on $\mathbb{E}^2 \times \{0\}$ and $\mathbb{E}^2 \times \{1\}$ for each $s \in [0,1]$, (this means it fixes the endpoints) and satisfies the property:

(*) $\forall s \in [0,1]$ the image set β_s of β under F_s is a geometric braid, and $\beta_0 = \beta$, $\beta_1 = \beta'$.

An isotopic deformation, or isotopy, is a homeomorphism for each s. This means F_s is continuous and invertible, and F_s^{-1} is continuous.

Since F_s is invertible, two points on the braid cannot map to the same point, so its strings cannot cross themselves. Of course that is taken care of by $(*)$, but $(*)$ is stronger since it also forbids "going back".

Notice that here it is the space $E^2 \times I$ that is being deformed. We could instead consider a different definition of equivalence, where we define continuous functions

$$G(t,s) = \{G_1(t,s), \dots, G_n(t,s)\}, \qquad G_j : I \times I \to \mathbb{C},$$

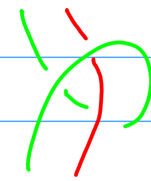
s.t. $G_j(t,0) = \underbrace{z_j(t)}_{\text{coords. of } \beta}, \quad G_j(t,1) = \underbrace{z_j'(t)}_{\text{coords of } \beta'}, \quad j=1,\dots,n$

with

$$\beta(s) = \{G_1(t,s), \dots, G_n(t,s)\} \text{ being a geometric braid } 0 \leq s \leq 1.$$

G is a <u>homotopy</u> and the braids are <u>homotopic</u>. We write $\beta \sim \beta'$.

The difference here is that homotopy only moves the braid continuously, not the space. However, Artin (1947a) proved that braid homotopy could always be extended to $E^2 \times I$, so that $\sim$ and $\approx$ are equivalent relations. (A similar result applies to arcs on n-manifolds: see J. Martin and D. Rolfsen, "Homotopic arcs are isotopic," Proc. of the AMS <u>19</u>, 1290 (1968).)

Now what if we relax (*) above, so that we allow strings to "go back" momentarily:  ?

In the same 1947 paper ["Theory of Braids," Annals of Math. **48**, 101 (1947)], Artin showed that this led to the same notion of equivalence. This is not all that surprising, since the homeomorphism property still implies that strings can't cross, but is good to know!

Hence, we will take $\underline{\approx}$ (isotopy) as our definition of equivalence of geometric braids.

We didn't quite show that $\approx$ defined an equivalence relation. But sort of obvious that $\approx$ satisfies

		Proof:
$\beta \approx \beta$	Reflexivity	$F_s = \mathrm{id}.$
$\beta \approx \beta' \Leftrightarrow \beta' \approx \beta$	Symmetry	$F_s \to F_{1-s}$
$\beta \approx \beta'$ & $\beta' \approx \beta'' \Rightarrow \beta \approx \beta''$	Transitivity	Compose F, G

$$F_s \qquad G_s \qquad H_s = \begin{cases} F_{2s}, & 0 \le s < \tfrac{1}{2} \\ F_1 \circ G_{2s-1}, & \tfrac{1}{2} \le s \le 1. \end{cases}$$

Next lecture we will see that braids form a groupoid.