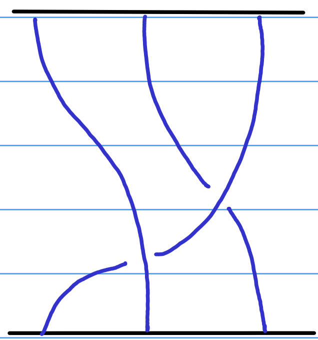
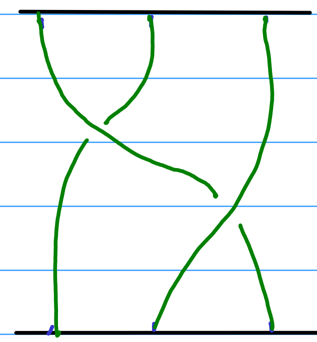
Braids – Lecture 3 : Artin Braid Groups

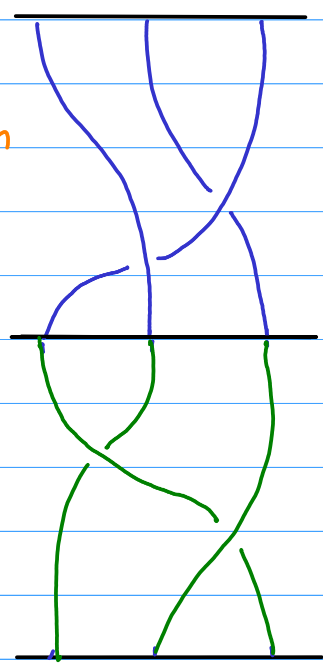
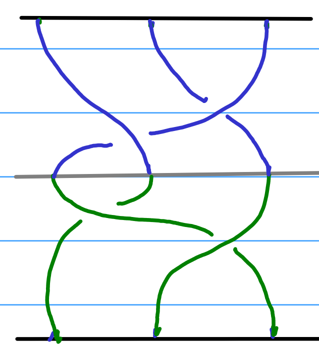
From now on we do not distinguish between a braid and its equivalence class: $=$ is the same as $\approx$.

Two braids that share the same base points can be composed together:

$\beta_1 =$  , $\beta_2 =$ 

product or composition rule

$\downarrow$

$\beta_1 \cdot \beta_2 =$  $=$ 

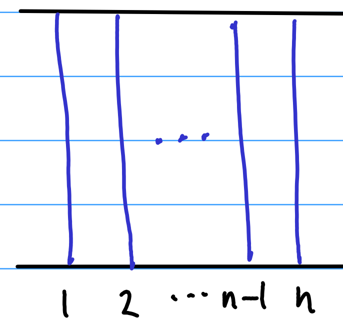
Simply stack the braids, join at the midplane, then squash to bring back to $E^2 \times I$.

It is obvious that the product is associative:

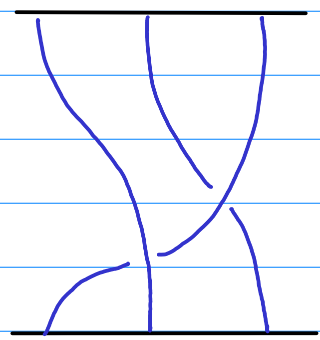
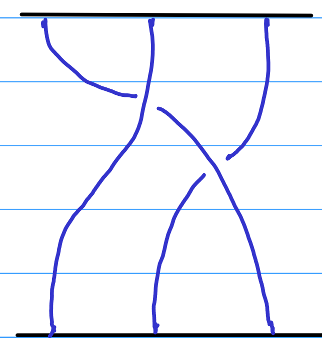
$$\beta_1 \cdot (\beta_2 \cdot \beta_3) = (\beta_1 \cdot \beta_2) \cdot \beta_3 .$$

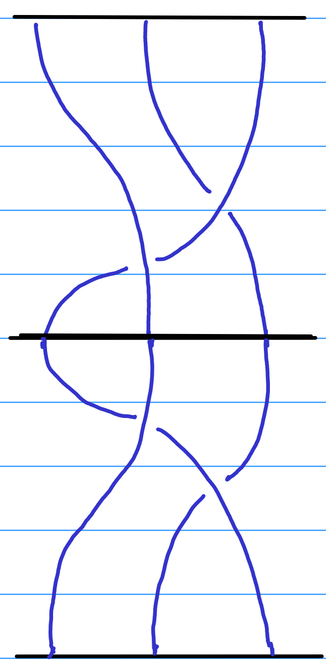
It is also closed over geometric braids.

The identity braid (or trivial) is

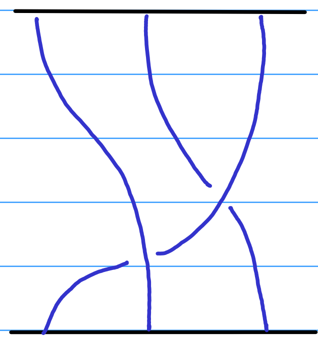
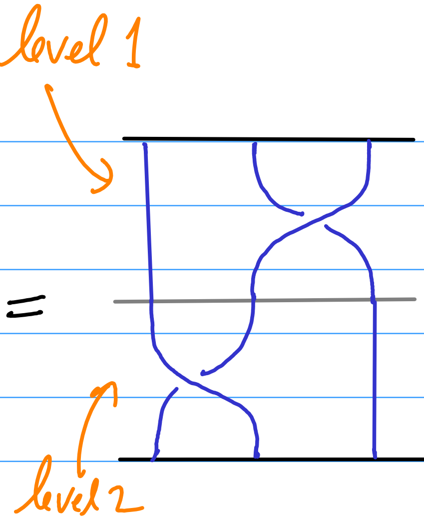
$e =$ 

Every braid has an inverse, obtained by "mirroring" the braid vertically:

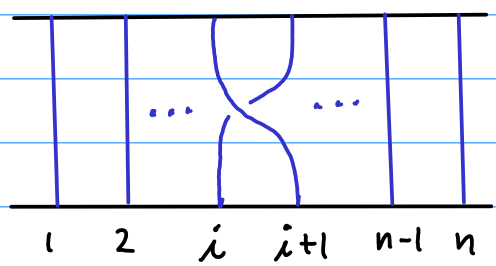
$\beta =$  , $\beta^{-1} =$ 

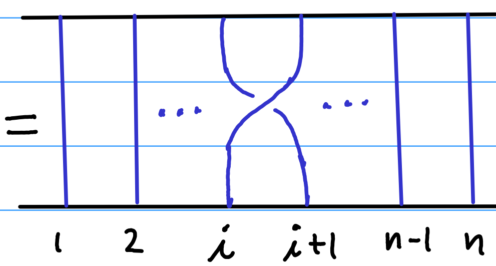
$\beta \cdot \beta^{-1} =$  $= e$.

We conclude that geometric n-braids with our composition rule form a group, the Artin braid group B_n.

$\beta =$  $=$ 

We can always draw a projection of a braid such that crossings occur at equally-spaced levels, with only one crossing per level. This suggest defining elementary braid operations or generators

$\sigma_i =$ 

, $\sigma_i^{-1} =$ 

Note that $\sigma_i \cdot \sigma_i^{-1} = e$ (or $= 1$), and $i \in \{1, \cdots, n-1\}$

It is intuitively obvious that any n-braid can be written as a product of the $n-1$ σ_i and their inverses. Hence, the elements σ_i, $1 \leq i \leq n-1$ generate the group B_n. B_n is a finitely-generated (countably) infinite group.

Example: β above:

$$\beta = \sigma_2^{-1} \sigma_1$$

$$\beta^{-1} = \sigma_1^{-1} \sigma_2$$

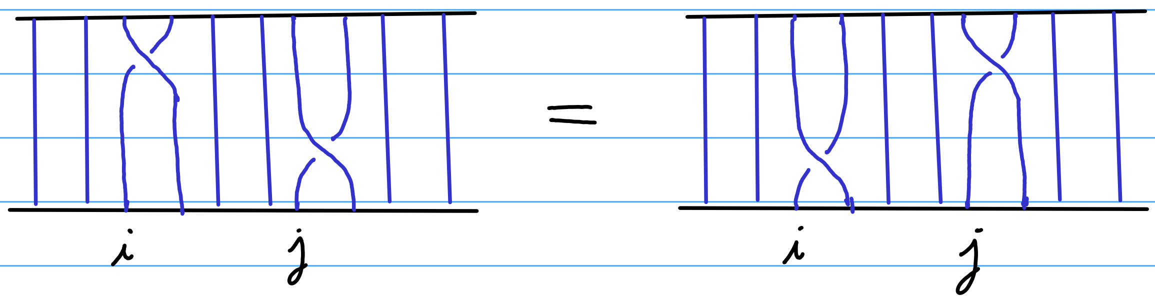
Write from left to right, working down the braid.

The generators σ_i obey relations of two types:

1) $$\sigma_i \sigma_j = \sigma_j \sigma_i \,, \qquad |i-j| > 1$$
$$1 \leqslant i,j \leqslant n-1$$

Note we usually leave out the "." from now on.

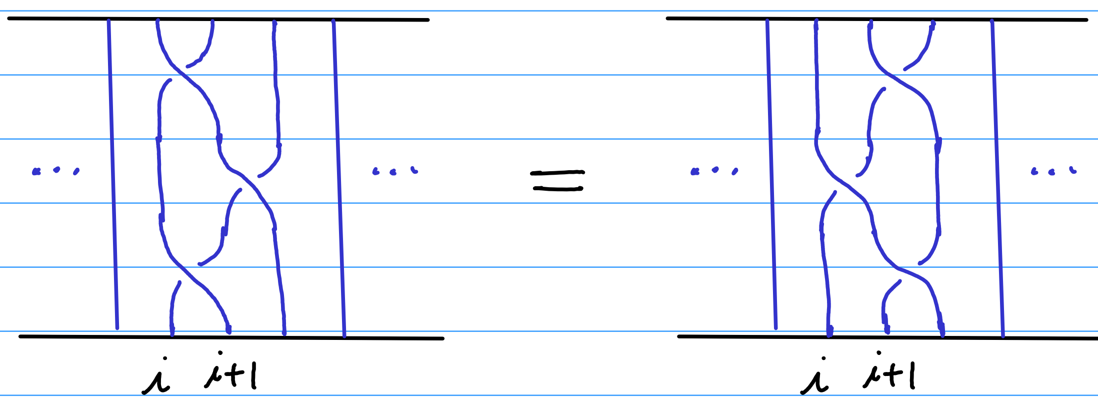
This says that non-adjacent generators commute:



The other type of relation is more interesting:

2) $$\sigma_i \sigma_{i+1} \sigma_i = \sigma_{i+1} \sigma_i \sigma_{i+1} \,, \quad 1 \leqslant i \leqslant n-2.$$

Better understood pictorially:



Stare at these and you will see they are the same.

The nontrivial fact is that these two types of relations are sufficient: there are no other relations among the generators that do not follow from 1+2 and elementary group operations (inverting, multiplying by σ_i^{-1}, etc.)

Example: $\sigma_i^{-1}\sigma_j = \sigma_j\sigma_i^{-1}$ follows from $\sigma_i\sigma_j = \sigma_j\sigma_i$
$|i-j|>1$

$$\cancel{\sigma_i^{-1}}\cancel{\sigma_i}\sigma_j\sigma_i^{-1} = \sigma_i^{-1}\sigma_j\cancel{\sigma_i}\cancel{\sigma_i^{-1}}$$

This means that $\{\sigma_1, \dots, \sigma_{n-1}\}$ together with 1) and 2) give a presentation for B_n.

A presentation implies a group isomorphism between B_n and the abstract braid group (or algebraic braid group) generated by $\{\sigma_1, \dots, \sigma_{n-1}\}$ with $\sigma_i\sigma_j = \sigma_j\sigma_i$, $|i-j|>1$, and $\sigma_i\sigma_j\sigma_i = \sigma_j\sigma_i\sigma_j$, $|i-j|=1$.

The fact that the above gives a presentation was proved by Artin in his 1925 paper. We shall have more to say about this later.

[Aside: a group generated by a set $\{x_1, \dots, x_n\}$ but where the x_i obey no special relations is called a free group.]