Braids – Lecture 4: Fundamental Groups

Now that we have a handle on braids as particle dances on $\mathbb{E}^2$, as geometric braids in $\mathbb{E}^2 \times I$, and as abstract braid groups, we will generalize the notion of a braid, so that we can define them on more general manifolds. For this we will need to introduce configuration spaces. But first we review fundamental groups, one of the most important notions of algebraic topology.

Let X be a topological space and $x_0 \in X$.
A loop with base point x_0 is a continuous function

w/o the base point restriction f gives a path.

$$f: I \to X \quad \text{with} \quad f(0) = f(1) = x_0.$$

Note that loops are oriented.

Two loops f and g with the same base point are considered equivalent if there is a homotopy between them:

$$F: I \times I \to X, \quad \begin{aligned} F(t,0) &= f(t) \\ F(t,1) &= g(t) \\ F(0,s) &= F(1,s) = x_0, \end{aligned}$$

with F continuous in its arguments.
The equivalence classes of loops are called homotopy classes.

We can define a product of loops $f * g$ by

$$(f*g)(t) = \begin{cases} f(2t), & 0 \leq t \leq 1/2; \\ g(2t-1), & 1/2 \leq t \leq 1. \end{cases}$$

Loop f, then loop g.

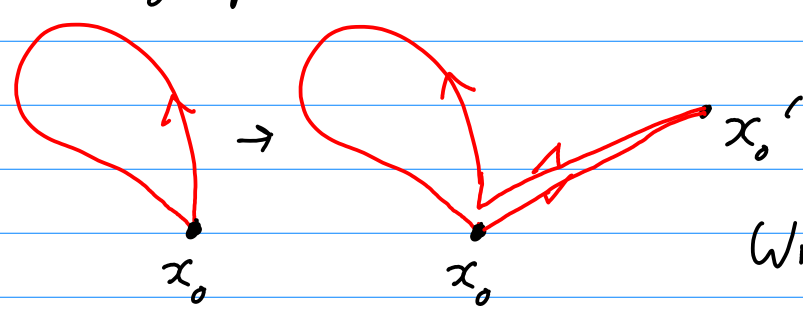
<u>Definition:</u> The set of homotopy classes of loops with base point x_0 is called the <u>Fundamental Group</u> of X based at x_0, and is denoted

$$\pi_1(X, x_0).$$

π_1 is also known as the <u>first homotopy group.</u>

The fact that π_1 is a group is easy to show. We have $(f^{-1})(t) = f(1-t)$, and identity $f(t) = x_0$.

For (path) connected spaces, π_1 is independent of the choice of base point x_0 (up to isomorphism). This is because we can always move the base point to another point x_0' by attaching paths:



This preserves the group composition law.

Write: $\pi_1(X, x_0) = \pi_1(X)$.

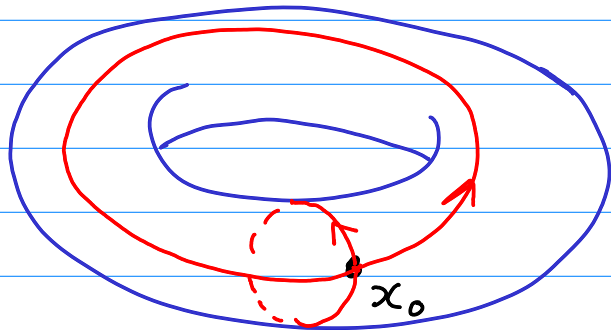
We can't do this if X has more than one component, since then x_0 and x_0' could be in different components.

<u>Simple example:</u> For $X = S^1$, the circle, closed loops go around the circle an integer number of times. Hence,

$$\pi_1(S) = \mathbb{Z}$$

← minus sign corresponds to going the other direction.

On the 2-torus $\pi_1(\mathbb{T}^2)$ is generated by two elements:

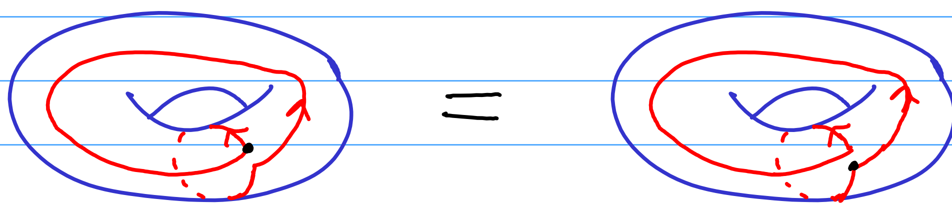


The fundamental group simply counts how many times we go around each direction of the torus:

$$\pi_1(\mathbb{T}^n, x_0) = \mathbb{Z}^n$$

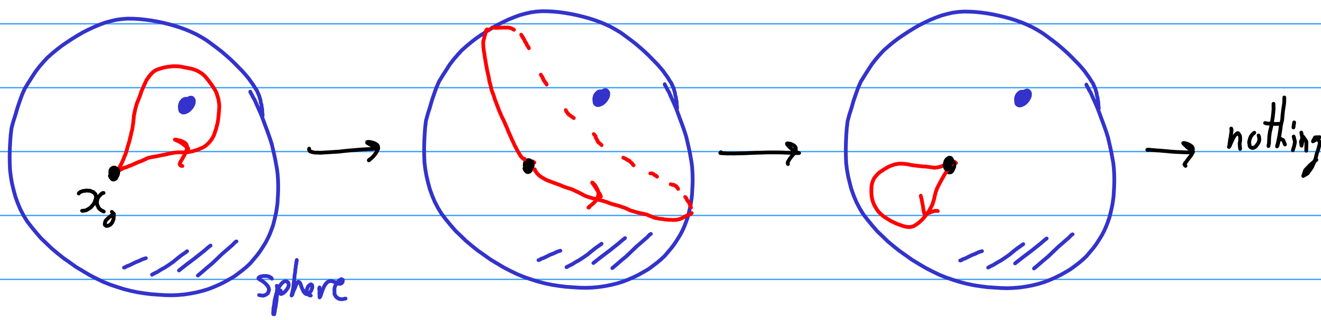
n-torus — group under addition

The fact that $\pi_1(\mathbb{T}^n, x_0)$ is Abelian is not entirely obvious! There are some nice movies on the web that show this: see Mathworld's fundamental group entry, or search on youtube.

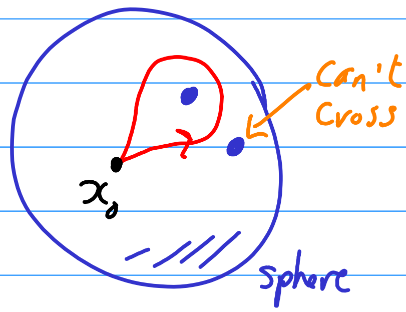


The sphere has a trivial fundamental group, since every loop is contractible to a point. (True for simply-connected spaces.)

The sphere with 1 puncture (or one boundary component) is the disk. But it also has a trivial π_1!

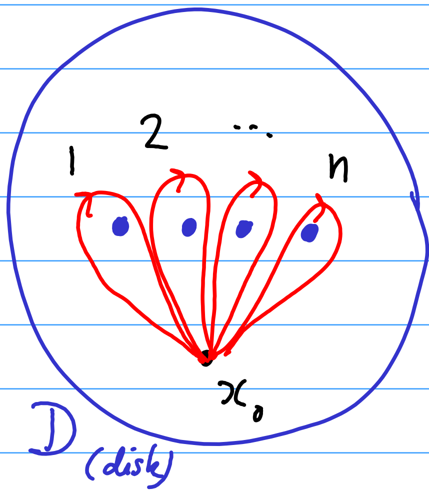


Need at least two punctures to prevent this:



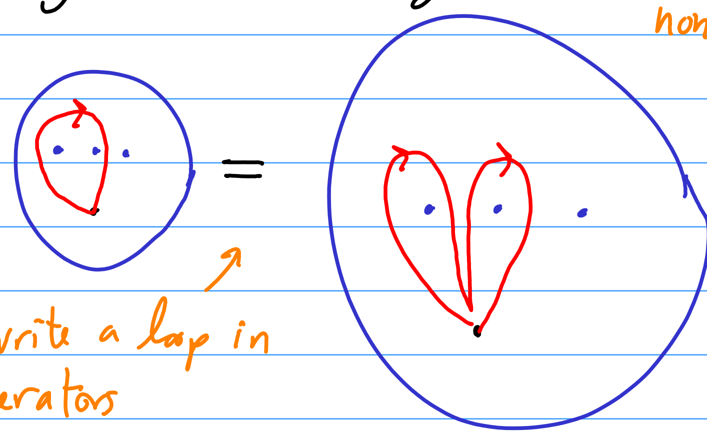
$\pi_1 = \mathbb{Z}$ in this case.

The sphere with 2 punctures is the same as the disk with 1 puncture.

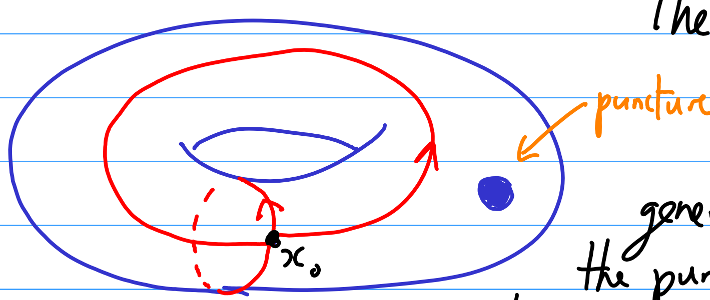


In the same manner the fundamental group for D with n punctures is $\mathbb{F}_n$, the free group with n generators.

non-Abelian!

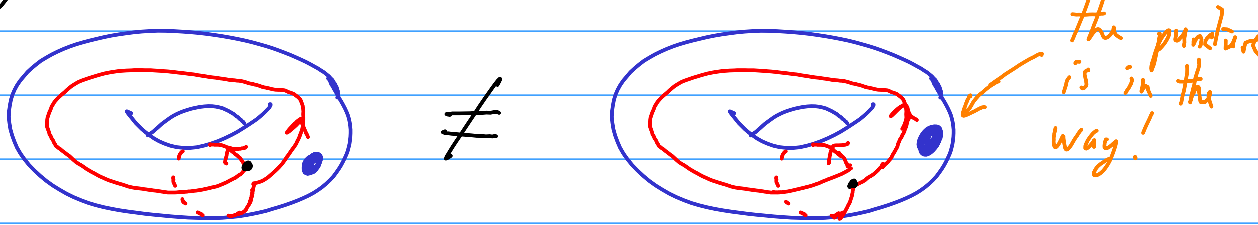


This is how we write a loop in terms of generators

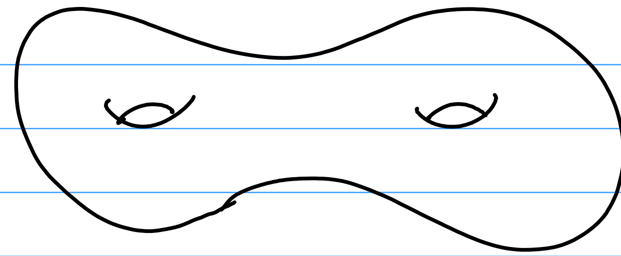


The torus with a puncture has a fundamental group generated by the same generators as the torus, but the puncture now prevents the two generators from commuting.

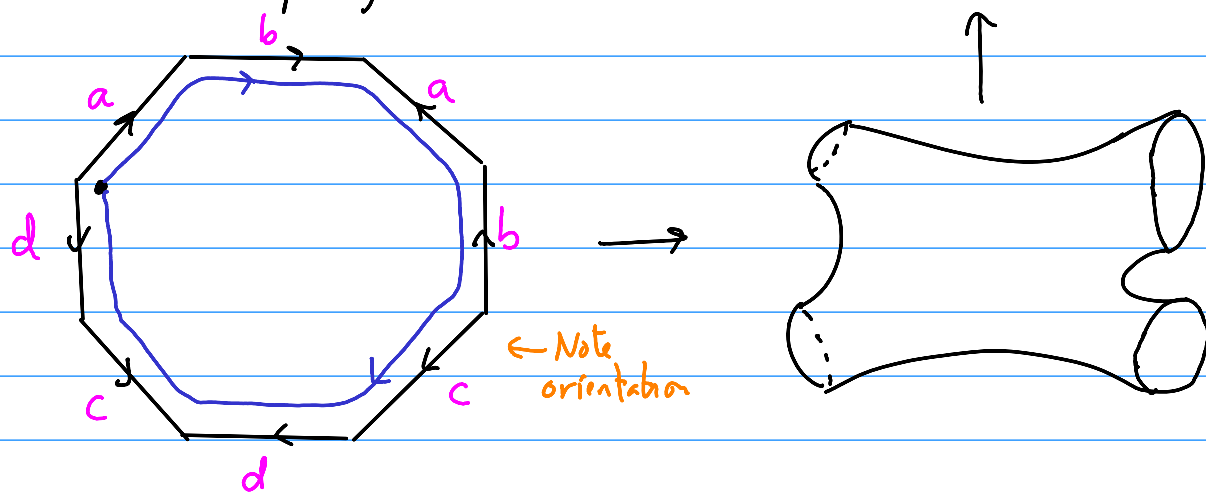
Hence, π_1 for this surface is $\mathbb{F}_2$.



The surface of genus 2:



can be cut and unfolded to make a "stop sign":



The path shown above is $aba^{-1}b^{-1}cdc^{-1}d^{-1} = 1$.

So $\pi_1 = \langle \underbrace{a,b,c,d}_{\text{4 generators}} \mid \underbrace{aba^{-1}b^{-1}cdc^{-1}d^{-1} = 1}_{\text{1 relation.}} \rangle$

Not a free group!

<u>Higher homotopy groups:</u> Let S^n be the n-sphere, and choose an arbitrary base point a in S^n.

We define $\pi_n(X, x_0)$ to be the set of homotopy classes of the maps

$$f: S^n \to X, \quad f(a) = x_0.$$

π_n is a group for $n \geq 1$. $\pi_0(X, x_0)$ has as many elements as there are path-connected components of X.

We shall not need higher homotopy groups much, but they will be needed briefly when dealing with fibrations.

A fibration is a map $f: E \to B$ between topological spaces that satisfies the homotopy lifting property.

What this means is that if we have a homotopy of a map into B, and we know how to lift the beginning of the homotopy into E, then we can lift the homotopy itself.

Locally, we can write $E = B \times F$, where F is the fibre space, and B is the base space.

Now comes the nice bit: given these definitions, there is a long exact sequence of homotopy groups

$$\cdots \to \pi_n(F) \to \pi_n(E) \to \pi_n(B) \to \pi_{n-1}(F) \to \cdots$$

$$\cdots \to \pi_0(F) \to \pi_0(E) \to \pi_0(B) \to 1.$$

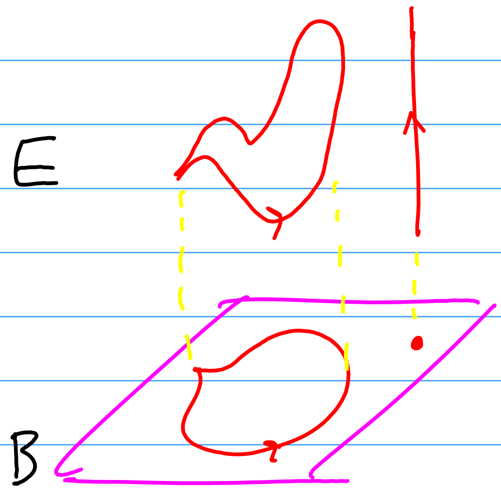
Each arrow is a group homomorphism, except for the last ones involving π_0. "Exact" means that in the subsequence

$$A \xrightarrow{a} B \xrightarrow{b} C$$

we have $\operatorname{im} a = \ker b$.

The sequence above will be useful in proving the presentation theorem for braid groups.

Justification for homotopy sequence:



$$\begin{array}{c} \pi_1(F) \\ \downarrow c \\ \pi_1(E) \\ \downarrow b \\ \pi_1(B) \\ \downarrow a \\ \pi_0(F) = 1 \end{array}$$

im c = loops on fibres injected in E

||

ker b = "along fibre"

im b $= \pi_1(B)$

surjective

|| by exactness

ker a $= \pi_1(B)$