Braids – Lecture 5: Configuration Spaces

Definition: Let M be a connected manifold of dimension ≥ 2. Then the configuration space for a set of $n \geq 1$ ordered points in M is

$$F_n(M) = \{(z_1, \cdots, z_n) \in M \times \cdots \times M \mid z_i \neq z_j \text{ for } i \neq j\}$$

[$F_n(M)$ is connected, and hence the homotopy groups of $F_n(M)$ are independent of the base point.]

The points are ordered. We can permute them by acting with the symmetric group Σ_n, which is simply the group of permutations on n symbols. The right-action $\mu: F_n(M) \times \Sigma_n \longrightarrow F_n(M)$ is defined by

$$\mu((z_1, \dots, z_n), \sigma) = (z_1, \cdots, z_n) \cdot \sigma = (z_{\underbrace{\sigma(1)}}, \cdots, z_{\underbrace{\sigma(n)}}), \quad \sigma \in \Sigma_n$$

The indices are permuted.

Σ_n acts freely on F_n, meaning that only the identity element in Σ_n fixes any element of F_n (this is because of the condition $z_i \neq z_j$.) We can form the orbit space

$$\mathcal{C}_n(M) = F_n(M) / \Sigma_n,$$

which is the same as dealing with unordered points. There is a natural projection $\tau: F_n \to \mathcal{C}_n$, mapping a point to its orbit.

The PURE BRAID GROUP $P_n(M)$ is then

$$P_n(M) = \pi_1(F_n(M), \vec{z}), \quad \vec{z} \in F_n(M)$$

and the BRAID GROUP is

$$B_n(M) = \pi_1(\mathcal{C}_n(M), \tau(\vec{z})),$$

where $\pi_1(X, x_0)$ is the fundamental group of X at the point x_0.

$\vec{z}$ is the base point, which since $\vec{z} \in F_n(M)$ gives us a set of n distinct points in M. A loop is a "dance" of points (never colliding) that each return to its initial position, or any permutation thereof for $B_n(M)$.

An example from robotics: Configuration spaces naturally arise when optimizing the motion of robots or AGVs (Automatic Guided Vehicles). The AGVs are moving around trying to perform some task, and in doing so must avoid collisions. If X is the factory floor containing some obstacles $\mathcal{O}$, then the natural configuration spaces to consider are

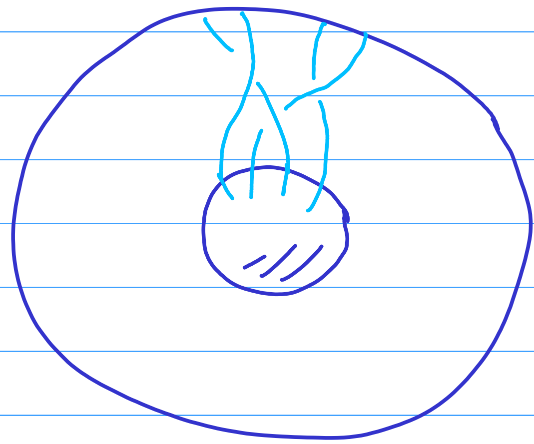
$F_n(X - \mathcal{O})$ for n different AGVs,

$\mathcal{C}_n(X - \mathcal{O})$ for n identical AGVs.

The π_1 of either of these gives us the possible paths the AGVs can follow.

Braid group on the sphere: $B_n(S^2)$ is still fairly easy to visualize.

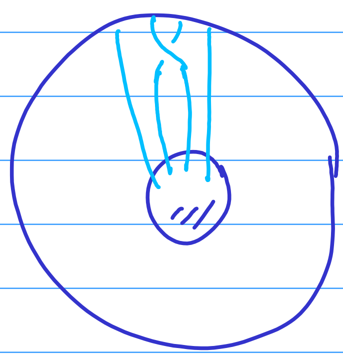
We imagine the braid as living between two concentric spheres:



Think of this as a cheap way of visualizing $S^2 \times I$.

We can still write $B_n(S^2)$ in terms of generators, δ_i, for $1 \leq i \leq n-1$.

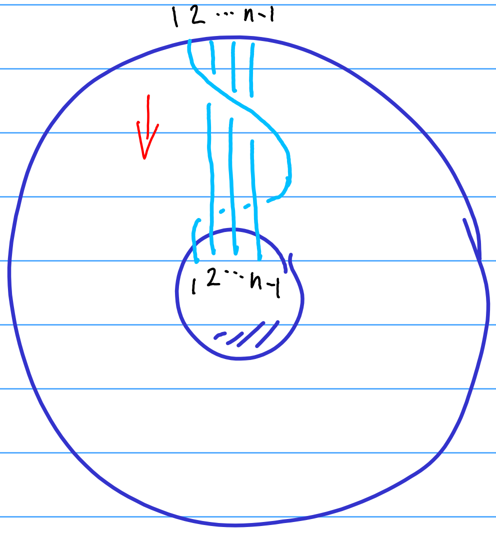
δ_2



The relations $\delta_i \delta_j = \delta_j \delta_i$, $|i-j| > 1$

$$\delta_i \delta_{i+1} \delta_i = \delta_{i+1} \delta_i \delta_{i+1}$$

are still obeyed, but there is now an extra relation:



$$\delta_1 \delta_2 \cdots \delta_{n-2} \delta_{n-1}^2 \delta_{n-2} \cdots \delta_2 \delta_1 = 1.$$