Braids – Lecture 6: The Presentation Theorem

Theorem: The Artin braid group B_n can be canonically identified with the fundamental group

$$B_n(\mathbb{E}^2) = \pi_1(\mathcal{C}_n(\mathbb{E}^2), \tau(\vec{z})).$$

We shall use the two descriptions interchangeably.

Let's establish the braid group sequence:

permutation homomorphism

Is it true if we replace $\mathbb{E}^2 \to M$?

$$1 \xrightarrow{a} P_n(\mathbb{E}^2) \xrightarrow{\tau_*} B_n(\mathbb{E}^2) \xrightarrow{p} \Sigma_n \xrightarrow{b} 1$$

is a short exact sequence, where τ_* is the group homomorphism induced by the orbit projection $\tau: F_n \to \mathcal{C}_n$. The homomorphism p takes a braid into its permutation.

$$\begin{aligned}
\ker \tau_* &= \text{the identity pure braid} \\
&= \operatorname{im} a. \\
\operatorname{im} \tau_* &= \text{braids that don't permute the points} \\
&= \ker p. \\
\operatorname{im} p &= \text{all permutations} \\
&= \ker b.
\end{aligned}$$

So the sequence is exact. Note that the homomorphisms a and b force $\ker \tau_* = 1$ and $\operatorname{im} p = \Sigma_n$, so they do play a role.

We can rewrite the braid group sequence for a manifold M as

$$1 \to \pi_1(F_n(M)) \xrightarrow{\tau_*} \pi_1(\mathcal{C}_n(M)) \xrightarrow{p} \Sigma_n \to 1$$

Since the map $\tau : F_n(M) \to \mathcal{C}_n(M)$ is an $n!$-fold covering map, we have a fibration (discussed in earlier lecture)

$$\tau : E \to B$$

$E = F_n(M)$
$B = \mathcal{C}_n(M)$
$F = \Sigma_n$ ← fibre

Hence, we have the long exact sequence

$$\cdots \to \pi_2(\Sigma_n) \to \pi_2(F_n(M)) \to \pi_2(\mathcal{C}_n(M)) \to$$

$$\to \pi_1(\Sigma_n) \to \pi_1(F_n(M)) \to \pi_1(\mathcal{C}_n(M))$$

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$$\to \underbrace{\pi_0(\Sigma_n)}_{\Sigma_n} \to \underbrace{\pi_0(F_n(M))}_{1,\ \text{since } F_n(M) \text{ is path-connected}} \to \pi_0(\mathcal{C}_n(M)) \to 1$$

This gives us the braid group sequence as a subsequence, but in fact there is nothing more.

Theorem: (Fadell & Neuwirth): $\pi_i(F_n(\mathbb{E}^2)) = 1$, $i \geq 2$.

See Hansen or Birman for a proof.

Not true for $M \neq \mathbb{E}^2$.
Example: $\pi_2(F_1(S^2)) = \pi_2(S^2) = \mathbb{Z}$
[Thanks to Joel for spotting this!]

This turns out to also imply that $B_n(\mathbb{E}^2)$ has no nontrivial elements of finite-order.

We now discuss the proof of the presentation theorem for the braid group.

Theorem: Let $\tilde{B}_n$ denote the abstract braid group

$$\tilde{B}_n = \left\langle \{\tilde{\sigma}_1, \dots, \tilde{\sigma}_{n-1}\} \;\middle|\; \begin{array}{ll} \tilde{\sigma}_i \cdot \tilde{\sigma}_j = \tilde{\sigma}_j \cdot \tilde{\sigma}_i \,, & |i-j|>1 \\ \tilde{\sigma}_i \tilde{\sigma}_{i+1} \tilde{\sigma}_i = \tilde{\sigma}_{i+1} \tilde{\sigma}_i \tilde{\sigma}_{i+1} \,, & i = 1, \dots, n-2 \end{array} \right\rangle$$

Then $\tilde{B}_n \cong B_n(\mathbb{E}^2)$.

The proof we sketch is due to Fadell and van Buskirk. Clearly, there is a homomorphism

$$\iota_n : \tilde{B}_n \longrightarrow B_n \qquad \text{where } B_n \equiv B_n(\mathbb{E}^2),$$

since we know the relations are "geometrically" correct. For B_n, we have

$$1 \longrightarrow P_n \xrightarrow{\tau_*} B_n \xrightarrow{p} \Sigma_n \longrightarrow 1$$

For $\tilde{B}_n$, we have a natural surjective homomorphism

$\tilde{p} : \tilde{B}_n \to \Sigma_n$ defined by mapping $\tilde{\sigma}_i$ onto the transposition $(i, i+1) \in \Sigma_n$.

Again, the homomorphism is well-defined, and since the transpositions generate Σ_n, it is surjective.

Let $\tilde{P}_n = \ker \tilde{p}$, and $\tilde{\tau} : \tilde{P}_n \to \tilde{B}_n$ the inclusion homomorphism.

For the abstract group we then have the short exact sequence

$$1 \longrightarrow \widetilde{P}_n \xrightarrow{\tilde{\tau}} \widetilde{B}_n \xrightarrow{\tilde{p}} \Sigma_n \longrightarrow 1$$

Thus we get the commutative diagram with exact rows:

$$\begin{tikzcd}
1 \arrow[r] \arrow[d, "1"'] & \widetilde{P}_n \arrow[r, "\tilde{\tau}"] \arrow[d, "c_n'"'] & \widetilde{B}_n \arrow[r, "\tilde{p}"] \arrow[d, "c_n"] & \Sigma_n \arrow[r] \arrow[d, "1"] & 1 \arrow[d, "1"] \\
1 \arrow[r] & P_n \arrow[r, "\tau"] & B_n \arrow[r, "p"] & \Sigma_n \arrow[r] & 1
\end{tikzcd}$$

where c_n' is the restriction of c_n to the subgroup $\widetilde{P}_n$ in $\widetilde{B}_n$.

Vertically, we have 3 bijective maps (1). The 5-lemma says that if the first and last two maps are bijective, then the central map is bijective.

Hence, c_n is bijective iff c_n' is bijective.

This means that we have reduced the presentation theorem for B_n to the problem of finding a presentation for P_n!

Is this better? Well, sort of. Because P_n leaves strings unpermuted, it is better suited to induction since we can peel strings off in a nice way. More on this next time...