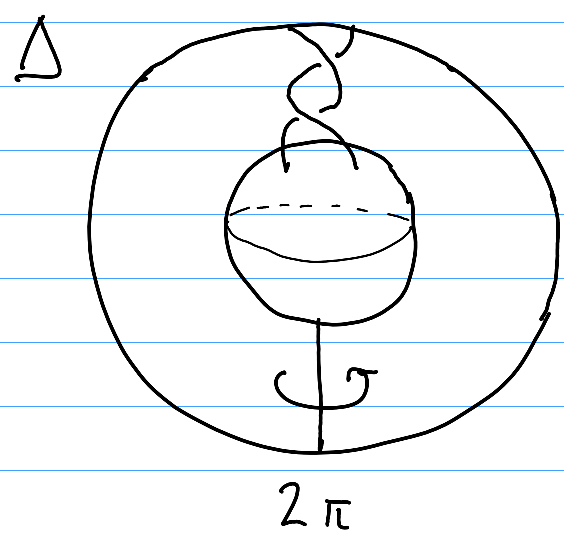
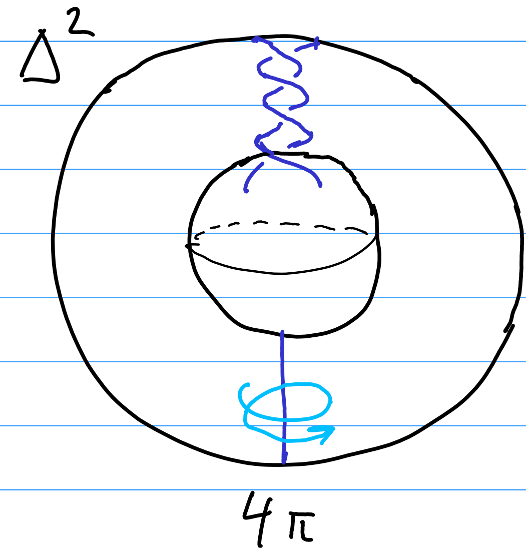
Braids Lecture 9: The Dirac String Trick

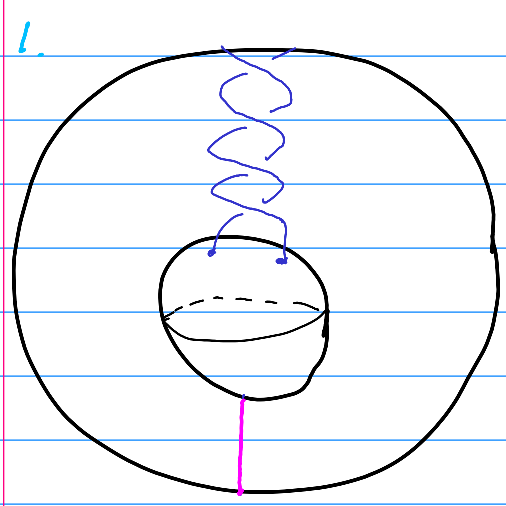
Recall the braid group on the sphere, $B_n(S^2)$, which we can represent as strings between two concentric spheres. The Dirac braid Δ consists of a twist of two strings in $B_3(S^2)$:

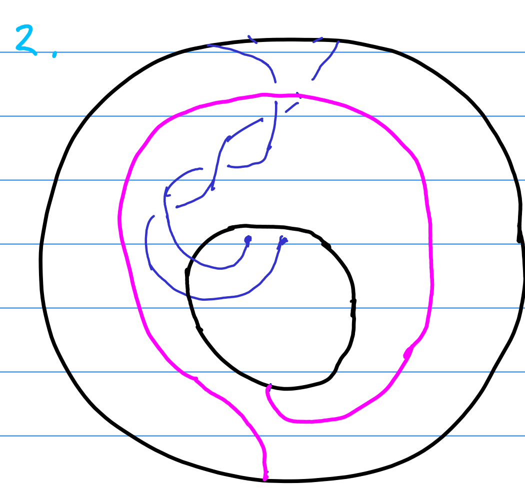




The claim is that $\Delta^m = 1$, m even, and Δ^m is nontrivial for m odd.

Show pictorially for Δ^2:





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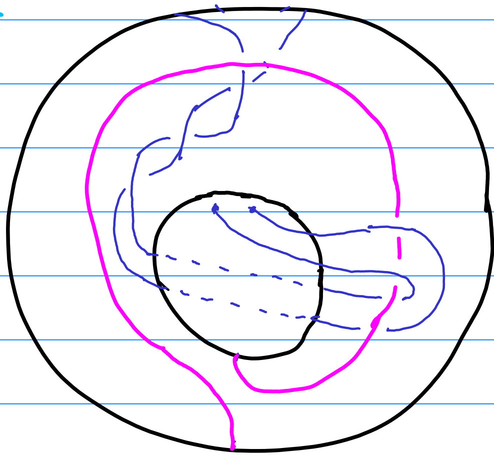

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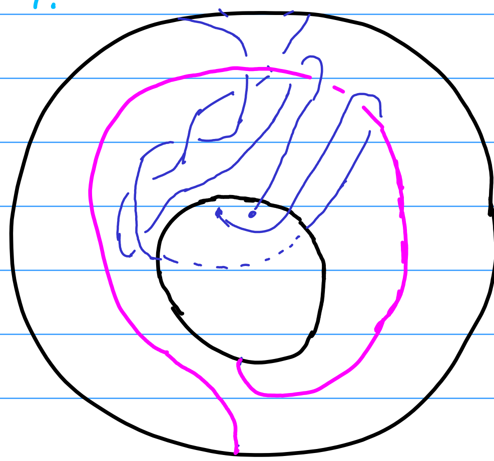

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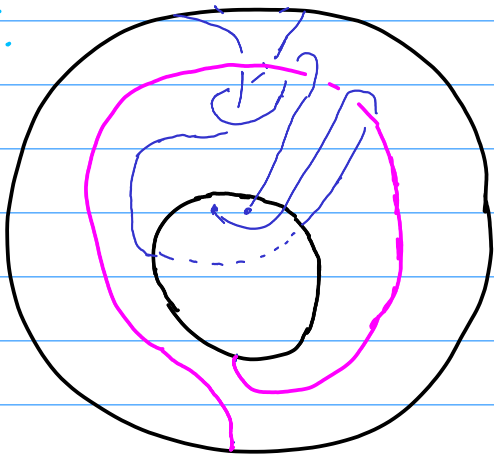

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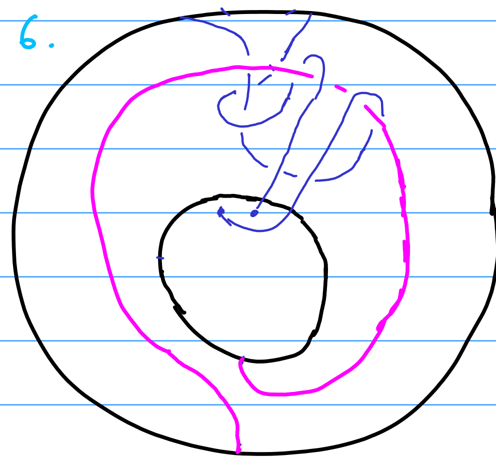

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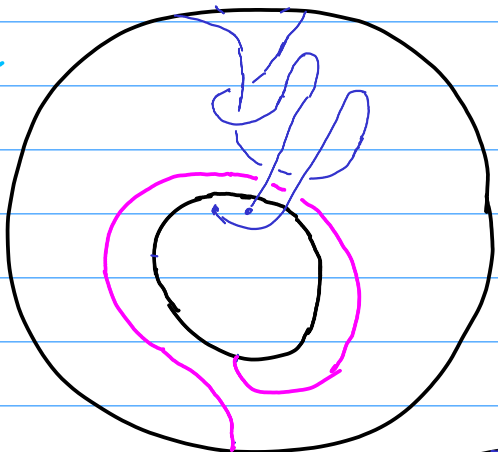

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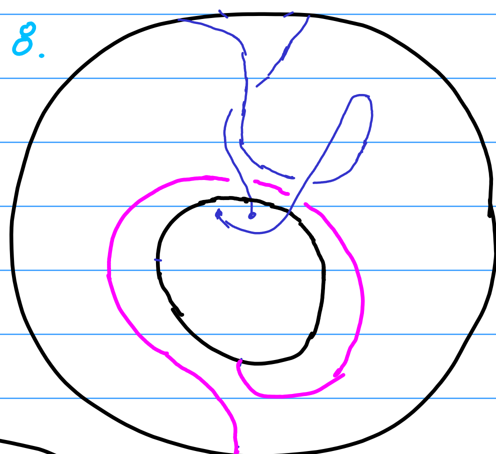

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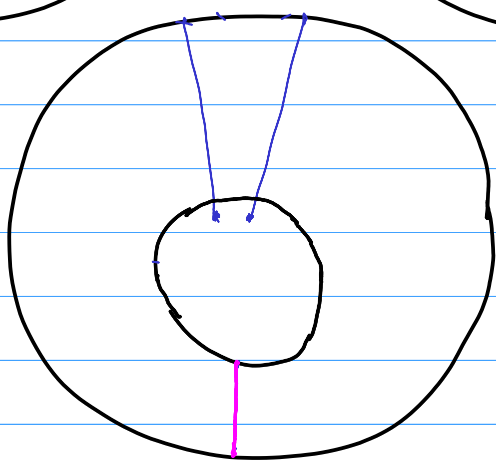
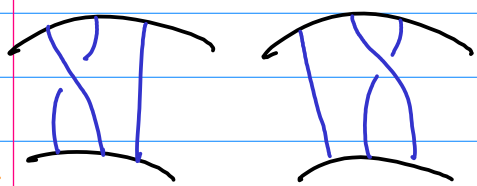


<u>An algebraic proof:</u> The relations in $B_3(S^2)$ are

Auroux gives a nice proof of presentation in his notes (lect. 5)



δ_1 δ_2

(i) $\delta_1\delta_2\delta_1 = \delta_2\delta_1\delta_2$

(ii) $\delta_1\delta_2^2\delta_1 = 1$

Assume this gives a presentation of $B_3(S^2)$.

With these generators we can write $\Delta = \delta_2^2$. If $\Delta = 1$, then we get a new relation $\delta_2^2 = 1 \Rightarrow \delta_1^2 = 1$ by (ii). This would give a new relation, independent of the two above.

Now, relation (ii) implies $\delta_2^2 = \delta_1^{-2}$. Hence, from (i),

$$\delta_1(\delta_2\delta_1\delta_2) = \delta_1(\delta_1\delta_2\delta_1) \Leftrightarrow (\delta_1\delta_2)^2 = \delta_1^2\delta_2\delta_1 = \delta_2^{-1}\delta_1$$

Multiply by $\delta_1\delta_2$ on the left: $(\delta_1\delta_2)^3 = \delta_1^2$.

Start again from (i), $(\delta_2\delta_1\delta_2)\delta_2 = (\delta_1\delta_2\delta_1)\delta_2$

$$\delta_2\delta_1\delta_2^2 = (\delta_1\delta_2)^2$$

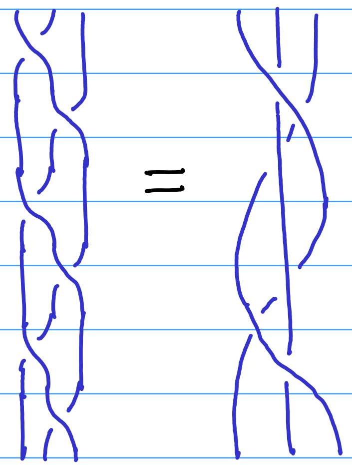
or $\delta_2\delta_1^{-1} = (\delta_1\delta_2)^2 \Longleftrightarrow \delta_2^2 = (\delta_1\delta_2)^3$. Δ^2

We conclude: $\delta_1^2 = \delta_2^2 = \delta_2^{-2}$, so that $\delta_2^4 = 1$.

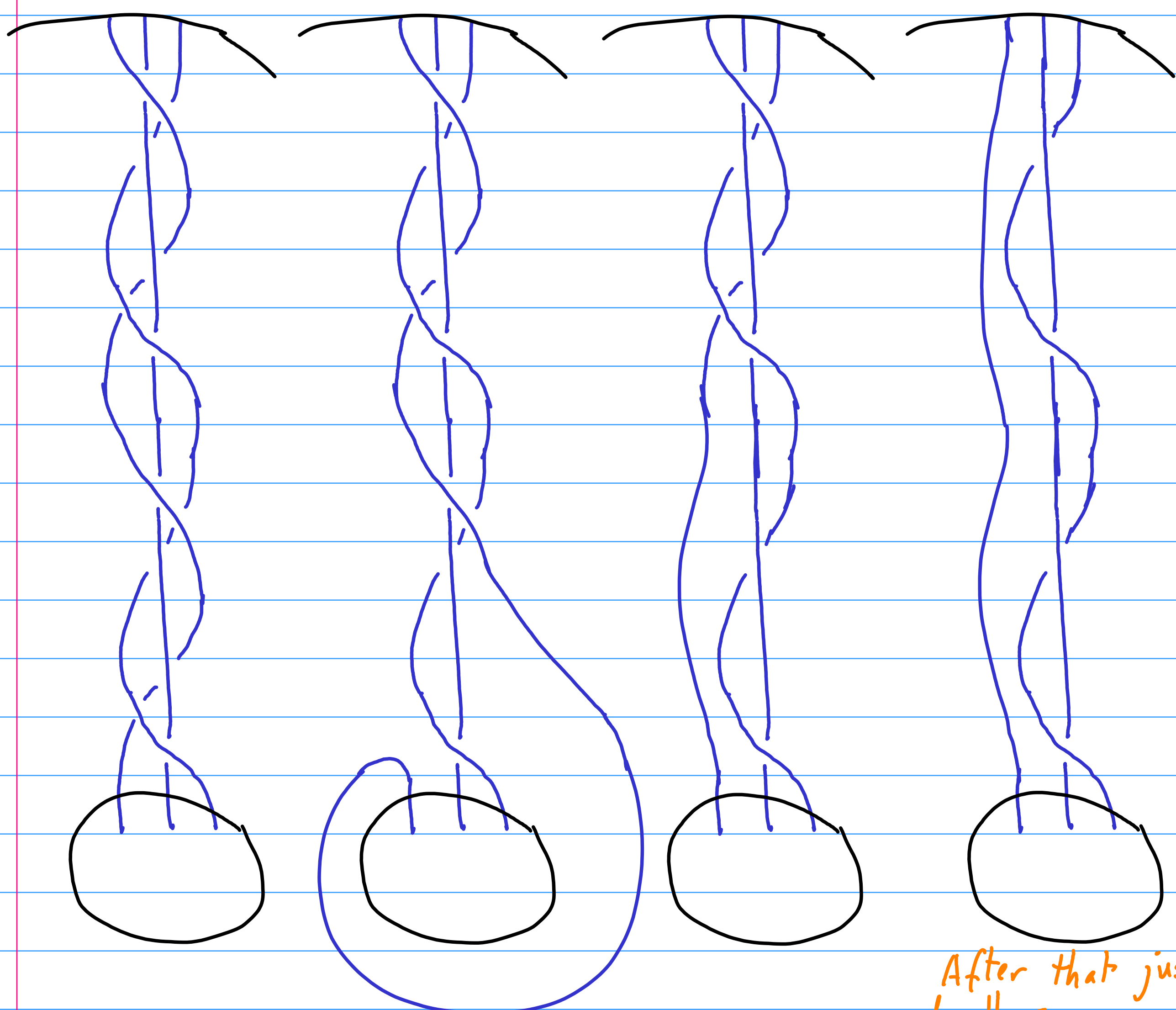
$$\therefore \quad \underline{\underline{\Delta^2 = 1.}} \qquad (\text{Also: } \Delta^{2m} = 1)$$

This is basically the Dirac string trick.

Note that $\delta_1^{\,2} = \delta_2^{\,2} = (\delta_1 \delta_2)^3$ is the full twist:



So the Dirac string trick can be summarized as: for the sphere, two full twists is the identity.



See Hansen for an algebraic-geometrical proof, based on $\pi_1(SO(3)) \cong \mathbb{Z}_2$.

After that just do the same as Dirac string trick

The group $B_3(S^2)$ is finite-order. Indeed, if we choose as generators

$$a = (\sigma_1\sigma_2)^2 , \qquad b = \sigma_1$$

Then a^3 = two full twists = 1

$$b^4 = \sigma_1^4 = 1.$$

$$b^{-1}ab = b^3 ab = \underbrace{\sigma_1^3(\sigma_1}_{1}\sigma_2\sigma_1\sigma_2)\sigma_1$$
$$= \sigma_2\sigma_1\sigma_2\sigma_1 = (\sigma_2\sigma_1)^2$$

But $a \cdot (\sigma_2\sigma_1)^2 = \sigma_1\sigma_2\underbrace{\sigma_1\sigma_2\,\sigma_2\sigma_1}_{1}\sigma_2\sigma_1 = \sigma_1\sigma_2^2\sigma_1 = 1$

so that $(\sigma_2\sigma_1)^2 = a^{-1} = a^2$.

$n=3$
$m=4$
$p=2$

$p^m \bmod n = 16 \bmod 3 = 1$.

← Mathworld notation for Metacyclic group.

Hence, $B_3(S^2) = \langle a, b \mid a^3 = b^4 = 1,\ b^{-1}ab = a^2 \rangle$

This is a ZS-metacyclic group of order 12.

Its elements are $\{1, b, b^2, b^3, a, ba, b^2a, b^3a, a^2, ba^2, b^2a^2, b^3a^2\}$.

Note that the relation $b^{-1}ab = a^2$ implies
$b^{-1}a^2b = (b^{-1}ab)^{-1} = (a^2)^{-1} = a$, so that $b^{-1}a^pb = a^q$.
This ensures that $\underbrace{\{1, a, a^2\}}_{L}$ is a cyclic normal subgroup of $B_3(S^2)$.

But also $B_3(S^2)/L$ is the group of cosets $[b] = ba^q$ and is also cyclic:

$$[b]^4 = ba^{q_1} ba^{q_2} ba^{q_3} ba^{q_4} = b^4 a^p = a^p = [1].$$

This is the definition of a metacyclic group G: it has a cyclic normal subgroup L $(= \{1, a, a^2\})$ such that G/L is also cyclic.

Half and Full Twists:

Of course, obviously $B_n(\mathbb{E}^2)$ $(n \geq 2)$ and $B_n(S^2)$ $(n \geq 4)$ are not finite groups. But the half-twist

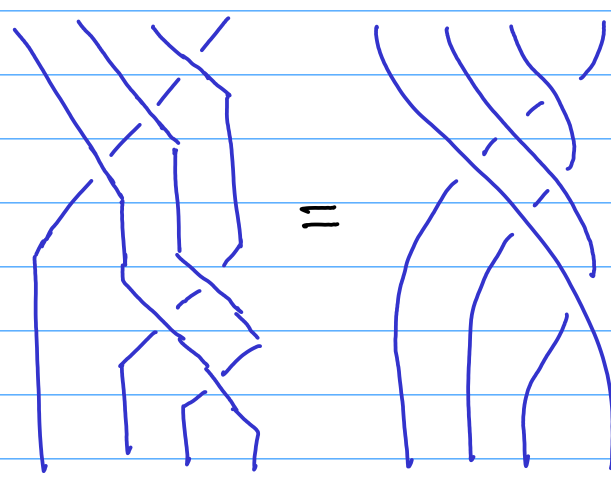
$$\Delta_n = (\sigma_{n-1}\sigma_{n-2}\cdots\sigma_1)(\sigma_{n-1}\sigma_{n-2}\cdots\sigma_2)\cdots(\sigma_{n-1}\sigma_{n-2})(\sigma_{n-1})$$

or Garside braid

and the full twist Δ_n^2 play a special role.
(More specifically, these are the positive half and full twists.)

$$\Delta_3 = \sigma_2\sigma_1\sigma_2 \quad , \quad \Delta_3^2 = \sigma_2\sigma_1\sigma_2\sigma_2\sigma_1\sigma_2 = \sigma_1\sigma_2\sigma_1\sigma_2\sigma_1\sigma_2 = (\sigma_1\sigma_2)^3$$

$$\Delta_4 = (\sigma_3\sigma_2\sigma_1)(\sigma_3\sigma_2)(\sigma_3)$$



(Note that this is unrelated the same as the Dirac braid Δ.)

$$\Delta_4^2 = \sigma_3\sigma_2\sigma_1\sigma_3\sigma_2\sigma_3\sigma_3\sigma_2\sigma_1\sigma_3\sigma_2\sigma_3$$

$$= \underline{\sigma_2\sigma_3\sigma_2}\sigma_1\sigma_2\sigma_3\sigma_3\sigma_2\sigma_3\sigma_1\sigma_2\sigma_3$$

$$= \sigma_2\sigma_3\underline{\sigma_1\sigma_2\sigma_1}\sigma_3\sigma_3\sigma_2\sigma_3\sigma_1\sigma_2\sigma_3$$

$$= \underline{\sigma_2\sigma_1}\underline{\sigma_2\sigma_3}\sigma_2\sigma_1\sigma_3\sigma_2\sigma_3\sigma_1\sigma_2\sigma_3$$

$$= \sigma_1\sigma_2\sigma_3\sigma_1\sigma_2\sigma_3\sigma_1\sigma_2\sigma_3\sigma_1\sigma_2\sigma_3 = (\sigma_1\sigma_2\sigma_3)^4$$

Recall the relations for $B_4(\mathbb{E}^2)$:

$$\sigma_1\sigma_3 = \sigma_3\sigma_1$$
$$\sigma_1\sigma_2\sigma_1 = \sigma_2\sigma_1\sigma_2$$
$$\sigma_2\sigma_3\sigma_2 = \sigma_3\sigma_2\sigma_3$$

In general, $\boxed{\Delta_n^2 = (\sigma_1\sigma_2\cdots\sigma_{n-1})^n}$ Positive full twist

$\Delta_n\sigma_i \neq \sigma_i\Delta_n$ since not the same "string".

$\Delta_n^2\sigma_i = \sigma_i\Delta_n^2$ Geometrically obvious!

Show for $n=4$, $i=2$:

$$\sigma_2^{-1}\Delta_4^2\sigma_2\Delta_4^{-2} = \sigma_2^{-1}\sigma_1\sigma_2\sigma_3\sigma_1\sigma_2\sigma_3\sigma_1\sigma_2\sigma_3\cancel{\sigma_1}\overbrace{\underline{\sigma_2\sigma_3\sigma_2}}^{\sigma_3\cancel{\sigma_2}\cancel{\sigma_3}}\cancel{\sigma_3^{-1}}\cancel{\sigma_2^{-1}}\cancel{\sigma_1^{-1}}\cancel{\sigma_3^{-1}}\sigma_2^{-1}\sigma_1^{-1}\sigma_3^{-1}\sigma_2^{-1}\sigma_1^{-1}\sigma_3^{-1}\sigma_2^{-1}\sigma_1^{-1}$$

$$= \sigma_2^{-1}\sigma_1\sigma_2\sigma_3\sigma_1\sigma_2\sigma_1\underbrace{\underline{\sigma_3\sigma_2\sigma_3}}_{\sigma_2\cancel{\sigma_3}\cancel{\sigma_2}}\cancel{\sigma_2^{-1}}\sigma_1^{-1}\cancel{\sigma_3^{-1}}\sigma_2^{-1}\sigma_1^{-1}\sigma_3^{-1}\sigma_2^{-1}\sigma_1^{-1}$$

$$= \sigma_2^{-1}\sigma_1\sigma_2\cancel{\sigma_3}\sigma_1\cancel{\sigma_2\sigma_1\sigma_2}\sigma_1\cancel{\sigma_1^{-1}\sigma_2^{-1}\sigma_1^{-1}}\cancel{\sigma_3^{-1}}\sigma_2^{-1}\sigma_1^{-1}$$

$$= \sigma_2^{-1}\sigma_2\sigma_1\sigma_2\sigma_2^{-1}\sigma_1^{-1} = 1.$$

So Δ_n^2, and hence Δ_n^{2m}, is in the centre — the subgroup of $B_n(\mathbb{E}^2)$ (or $B_n(S^2)$) whose elements commute with all the elements of $B_n(\mathbb{E}^2)$. In fact, $\underline{\Delta_n \text{ generates the centre}}$, an infinite cyclic subgroup in $B_n(\mathbb{E}^2)$ $(n \geq 3)$, order 2 for $B_n(S^2)$.
(Dirac)