Braids Lectures 10-12: Mapping Class Groups

Let $S = S_{g,b,n}$ denote a 2-manifold of genus g with b boundary components and n punctures.

$S_{0,0,0}$	sphere
$S_{1,0,0}$	torus
$S_{0,1,0}$	disc
$S_{0,1,n}$	disc with n punctures.

We follow the review of Birman & Brendle.

Let $\mathrm{Diff}^+(S)$ denote the group of all orientation-preserving diffeomorphisms of S

$\mathrm{Diff}^+(S)$ can be given the compact-open topology, making it into a topological group.

We allow diffeomorphisms in $\mathrm{Diff}^+(S)$ to permute the punctures, but they fix the boundari(es), ∂S, pointwise:

$$h|\partial S = \mathrm{id}.$$

As opposed to $h(\partial S) = \partial S$.

We write $\mathrm{Diff}^+(S_{g,b,\hat{n}})$ when the punctures are to be fixed pointwise.

$$h \in \mathrm{Diff}^+(S_{g,b,\hat{n}}) \Rightarrow h(p_i) = p_i, \; i=1,\dots,n.$$

Otherwise, $h \in \mathrm{Diff}^+(S_{g,b,n}) \Rightarrow h(\{p_1,\dots,p_n\}) = \{p_1,\dots,p_n\}$.

The Mapping class group $\mathcal{M} = \mathcal{M}_{g,b,n}$ of $S_{g,b,n}$ is

$$\mathcal{M}_{g,b,n} = \pi_0\left(\mathrm{Diff}^+(S_{g,b,n})\right)$$

Recall that $\pi_0(X)$ "counts" the path-connected components of X. Two diffeos are thus in the same homotopy class, and so define the same element in $\mathcal{M}$, if they can be continuously "deformed" into each other, in the sense of homotopy. We will see what this means a bit later.

Theorem: There are natural isomorphisms

punctures fixed

$$B_n(S_{0,1,0}) \cong \mathcal{M}_{0,1,n}, \quad P_n(S_{0,1,0}) \cong \mathcal{M}_{0,1,\hat{n}}$$

Intuitive picture: Choose $h \in \mathrm{Diff}^+(S_{0,1,n})$.

In general h is not isotopic to the identity, because we have to fix the punctures (and the boundary) when doing an isotopy.

This is the diff. between punctures & boundaries. →

Let $i : \mathrm{Diff}^+(S_{0,1,n}) \longrightarrow \mathrm{Diff}^+(S_{0,1,0})$ be the inclusion map, well-defined since $\mathrm{Diff}^+(S_{0,1,n})$ is a (closed) subgroup of $\mathrm{Diff}^+(S_{0,1,0})$. Then clearly $i(h)$ is isotopic to the identity, since we don't have to fix anything but the boundary of the disc. (this would not be true for $g>0$ or $b>0$!)

In fact, $\pi_0(\mathrm{Diff}^+(S_{0,1,0})) = 1$: every diffeo can be isotoped to the identity. [See homotopy in 2 pages.]

Let $h_t : \mathrm{Diff}^+(S_{0,1,0}) \to \mathrm{Diff}^+(S_{0,1,0})$ be the isotopy that takes $i(h)$ to the identity. $(h_0 = i(h), h_1 = 1)$

If the punctures in $S_{0,1,n}$ are at $(p_1, \cdots, p_n)$, then the n paths

$$(h_t(p_1), \cdots, h_t(p_n))$$

sweep out a braid in $S_{0,1,0} \times I$, and the equivalence class of this braid is the image of the mapping class $[h]$ in the braid group B_n.

The inverse isomorphism, from $B_n(S_{0,1,0})$ to $M_{0,1,n}$, is trickier. B & B take of starting with a braid and "flattening it" so that the n braid strings become non-intersecting simple arcs. Each of these arcs begins and ends at a base point. One then constructs a homeomorphish of the punctured disc to itself in such a way as the trace of the isotopy to the identity is the given set of n non-intersecting simple arcs.

My problem is with "non-intersecting". Can this be true? Does this mean pairwise or itself?

Another way to see the isotopy is to attach a rubber sheet at the top of the braid, and push the sheet down, letting the punctures deform the sheet. The deformation of the rubber sheet then gives an idea of the isotopy to the identity.

Now we give the proof of the isomorphism.

or $\mathbb{E}^2$

Proof: We begin by $P_n(S_{0,1,0}) \cong \mathcal{M}_{0,1,\hat{n}}$.

Facts:
- $\mathrm{Diff}^+(S_{0,1,0})$ is a topological group
  (compact-open topology: the open sets in $S_{0,1,0}$ are used to define open sets in the function space Diff^+.)
- $\mathrm{Diff}^+(S_{0,1,\hat{n}})$ is a closed subgroup of $\mathrm{Diff}^+(S_{0,1,0})$.
  ("closed" is easy to see: any diffeo in a sequence fixes the punctures, so limit must be in $\mathrm{Diff}^+(S_{0,1,\hat{n}})$.)

Define the evaluation map

configuration space

$$\mathcal{E}: \mathrm{Diff}^+(S_{0,1,0}) \longrightarrow F_n(S_{0,1,0})$$

by $\mathcal{E}(h) = (h(p_1), \dots, h(p_n))$.

$\mathcal{E}$ is continuous (w.r.t. the compact-open topology on $\mathrm{Diff}^+(S_{0,1,0})$ and the subspace topology for $F_n(S_{0,1,0}) \subset \mathbb{E}^2 \times \cdots \times \mathbb{E}^2$, or $\mathbb{C} \times \cdots \times \mathbb{C}$), and acts n-transitively on the disk, meaning that

If $(p_1, \dots, p_n)$ are n distinct points, and $(q_1, \dots, q_n)$ are n others, then there is an $h \in \mathrm{Diff}^+(S_{0,1,0})$ such that $h(p_i) = q_i$, $i = 1, \dots, n$.

If $\hat{h} \in \mathrm{Diff}^+(S_{0,1,\hat{n}})$, then

Base point for homotopies

$$(\hat{h}(p_1), \dots, \hat{h}(p_n)) = (p_1, \dots, p_n),$$

and if $h, h' \in \mathrm{Diff}^+(S_{0,1,0})$ with $\mathcal{E}(h) = \mathcal{E}(h')$, then h and h' are in the same left coset of $\mathrm{Diff}^+(S_{0,1,\hat{n}})$ in $\mathrm{Diff}^+(S_{0,1,0})$.

That is, $h\hat{h} = h'\hat{h}'$ for some $\hat{h}, \hat{h}' \in \mathrm{Diff}^+(S_{0,1,\hat{n}})$, since h and h' move the punctures in the same manner, and $\hat{h}$ and $\hat{h}'$ do not move the punctures.

With these observations, we conclude that $\mathcal{E}$ is a fibration with

$$1 \to \underset{\text{Fibre } F}{\mathrm{Diff}^+(S_{0,1,\hat{n}})} \xrightarrow{i} \underset{\text{Total space } E}{\mathrm{Diff}^+(S_{0,1,0})} \xrightarrow{\mathcal{E}} \underset{\text{Base space } B}{F_n(S_{0,1,0})} \to 1$$

[Follows from the "bundle structure theorem, see Steenrod.]

ker $\mathcal{E}$ consists of those $h \in \mathrm{Diff}^+(S_{0,1,0})$ that leave the punctures alone, but these are exactly elements $i(\hat{h})$ with $\hat{h} \in \mathrm{Diff}^+(S_{0,1,\hat{n}})$, so ker $\mathcal{E}$ = im i.

Now, yet again (!), we use the homotopy sequence corresponding to the above,

$$\cdots \to \pi_1(\mathrm{Diff}^+(S_{0,1,0})) \xrightarrow{\mathcal{E}_*} \overbrace{\pi_1(F_n(S_{0,1,0}))}^{\cong P_n(S_{0,1,0})} \xrightarrow{\partial_*} \underbrace{\pi_0(\mathrm{Diff}^+(S_{0,1,\hat{n}}))}_{\mathcal{M}_{0,1,\hat{n}}}$$

$$\xrightarrow{i_*} \underset{1}{\pi_0(\mathrm{Diff}^+(S_{0,1,0}))} \to \cdots$$

$\pi_0(\mathrm{Diff}^+(S_{0,1,0})) = 1$, as discussed above.

$\pi_1(\mathrm{Diff}^+(S_{0,1,0})) = 1$ as well, so

$$1 \xrightarrow{\varepsilon_*} P_n(S_{0,1,0}) \xrightarrow{\partial_*} \mathcal{M}_{0,1,\hat{n}} \xrightarrow{i_*} 1$$

$$\operatorname{im} \varepsilon_* = 1 = \ker \partial_* \leftarrow \text{exactness}$$

$$\operatorname{im} \partial_* = \ker i_* = \mathcal{M}_{0,1,\hat{n}}$$

$\therefore$ ∂_* is an isomorphism.

This proves the theorem for $P_n(S_{0,1,0})$. For $B_n(S_{0,1,0})$, observe that we can construct the short exact sequence

just as we did for the braid groups:

$$\begin{tikzcd} 1 \arrow[r] & \mathcal{M}_{0,1,\hat{n}} \arrow[r] \arrow[d] & \mathcal{M}_{0,1,n} \arrow[r] \arrow[d] & \Sigma_n \arrow[r] \arrow[d] & 1 \\ 1 \arrow[r] & P_n \arrow[r] & B_n \arrow[r] & \Sigma_n \arrow[r] & 1 \end{tikzcd}$$

The bijectivity of the central homorphism follows from the five lemma.

"compactly supported" → $\mathrm{Homeo}^+_c(\mathbb{E}^2)$ is contractible:

Given h, let

$\varphi_t(h) : z \mapsto t h\left(\frac{z}{t}\right)$.

$\varphi_t(h)$ tends continuously to id as $t : 1 \to 0$: $w = \frac{z}{t}$

$$\left|z - t h\left(\tfrac{z}{t}\right)\right| = t|w - h(w)| \leq \underbrace{t|id - h|_{L^\infty}}_{\text{bounded since } h \text{ has compact support.}} \to 0$$

See Hansen p.36 for disc. "Alexander trick".

Also F+M p.56

Note that sometimes we jump back and forth between Homeo^+ and Diff^+. This is of no consequence since the two MCGs equivalent: see Farb & Margalit "A Primer on MCGs", v. 2.95 p. 42 & Appendix A.

Relaxing the requirement of fixing ∂S pointwise:

So far we assummed that elements of $\mathrm{Diff}^+(S)$ fixed ∂S pointwise. If we relax this to fixing ∂S setwise, then the proof above is unchanged until we get to the homotopy sequence, Now we have

$$\pi_1(\mathrm{Diff}^+(S_{0,1,0},\underbrace{\{\partial S\}}_{\text{what is being fixed}}))=\mathbb{Z}$$ if we fix $\{\partial S\}$ rather than ∂S.

We can justify this as follows: recall that we are dealing with a closed path in $\mathrm{Diff}^+(S_{0,1,0},\{\partial S\})$ that begins and ends at id (basepoint). But to retract this loop to the identity, we must keep the basepoints fixed

$$\pi_1(\mathrm{Diff}^+(S_{0,1,0},\{\partial S\})), \quad h: S_{0,1,0}\circlearrowleft$$

$R(\theta)$ = rotation of disc by θ.

Loop: $h_t(z) = R(2\pi m t)z,\ t\in I,\ m\in\mathbb{Z},$

This is a closed loop since $h_0(z)=z=R(2\pi m)z=h_1(z)$.
Now try to define an homotopy to the trivial loop:

$$F(s,t) \text{ such that } F(1,t)=h_t,\quad F(0,t)=id.$$

Try something like $F(s,t)(\theta)=R(2\pi m s t)$

Oops! Not a closed loop for noninteger s. s cannot change continuously.

This heuristically shows why

$$\pi_1(\mathrm{Diff}^+(S_{0,1,0}),\{\partial S\})=\mathbb{Z}.$$

If ∂S is fixed pointwise, can't come back to identity if there is a rotation! So $\pi_1=1$.

Recall the homotopy sequence:

$$\cdots \longrightarrow \pi_1(\mathrm{Diff}^+(S_{0,1,0}),\{\partial S\}) \xrightarrow{\varepsilon_*} P_n(S_{0,1,0}) \xrightarrow{\partial_*} \mathcal{M}_{0,1,\hat{n}} \xrightarrow{i_*} \mathcal{M}_{0,1,0} = 1$$

Now this is $\mathbb{Z}$!

ε_* maps the nontrivial closed loops in $\pi_1(\mathrm{Diff}^+(S_{0,1,0}),\{\partial S\})$, which consist of full rotations of the boundary to elements of the pure braid group. Clearly,

See next two pages for more details!

$$\varepsilon_* : m \longrightarrow \Delta_n^{2m}$$

rotate boundary by $2\pi m$

m full twists in P_n!

Conclude: $\operatorname{im} \varepsilon_* = \langle \Delta_n^2 \rangle = \ker \partial_*$.

center of P_n

Can "unwind" full twists to get id in $\mathcal{M}_{0,1,\hat{n}}$.

Conclude:

$$\boxed{\mathcal{M}_{0,1,\hat{n}} = P_n(S_{0,1,0}) / \langle \Delta_n^2 \rangle}$$

when we allow the boundary to move in $\mathrm{Diff}^+(S_{0,1,0},\{\partial S\})$.

Again, the same result follows for $\mathcal{M}_{0,1,n} = B_n(S_{0,1,0}) / \langle \Delta_n^2 \rangle$.

In the next two pages we'll justify the exact sequence by looking at $\mathcal{E}_*$.

Recall the evaluation map

configuration space

$$\mathcal{E}: \mathrm{Diff}^+(S_{0,1,0}) \longrightarrow F_n(S_{0,1,0})$$

defined by $\mathcal{E}(h) = (h(p_1), \ldots, h(p_n))$

and the homotopy sequence:

$$\cdots \to \underbrace{\pi_1(\mathrm{Diff}^+(S_{0,1,0}), \{\partial S\})}_{\mathbb{Z}} \xrightarrow{\mathcal{E}_*} P_n(S_{0,1,0}) \xrightarrow{\partial_*} \mathcal{M}_{0,1,\hat{n}}$$

$$\xrightarrow{i_*} \mathcal{M}_{0,1,0} = 1$$

When ∂S is fixed pointwise π_1 does not contain rotations.

A loop in $\mathrm{Diff}^+(S_{0,1,0})$ evaluates to a braid by following only the punctures under the action of the diffeos.

If $h_t \in \pi_1(\mathrm{Diff}^+(S_{0,1,0}), \{\partial S\})$ rotates the boundary $m \in \mathbb{Z}$ times, choose a loop

$$h_t: z \longmapsto R(2\pi m t) z \qquad \begin{matrix} t \in I \\ m \in \mathbb{Z} \end{matrix}$$

where $R(\theta)$ is a rotation of the disc by θ. Then clearly $\mathcal{E}_*(h_t) = \Delta_n^{2m}$, where Δ_n^2 is the full twist.

The question is whether we could have picked a different element of $\pi_1(\mathrm{Diff}^+(S_{0,1,0}), \{\partial S\})$, one that did something else the the punctures. Recall that

$$\partial_* : \underbrace{\pi_1(F_n(S_{0,1,0}))}_{P_n(S_{0,1,0})} \longrightarrow \underbrace{\pi_0(\mathrm{Diff}^+(S_{0,1,\hat{n}}, \{\partial S\}))}_{\mathcal{M}_{0,1,\hat{n}}}.$$

By exactness, $\ker \partial_* = \operatorname{im} \varepsilon_*$, so $\varepsilon_*(h_t)$ must map to the trivial element of $\mathcal{M}_{0,1,\hat{n}}$, that is, the isotopy class of the identity. This must correspond to $\langle \Delta_n^2 \rangle$, since these are the identity by rotating the boundary.

Punctures vs Pointwise-fixed Boundaries:

In fact this question of whether we can rotate the boundaries is really what distinguishes punctures from boundaries. If the boundaries are fixed pointwise, then there is an exact sequence

$$1 \to \mathbb{Z}^b \longrightarrow \mathcal{M}_{g,b,0} \xrightarrow{i_*} \mathcal{M}_{g,0,b} \longrightarrow 1$$

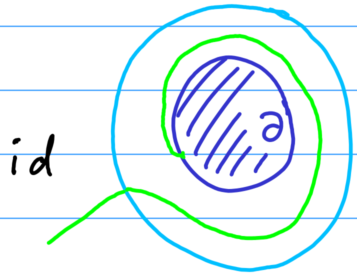
↑ b boundaries ↑ b punctures

True for Homeo^+. Not sure about Diff^+.

See Auroux p. 44.

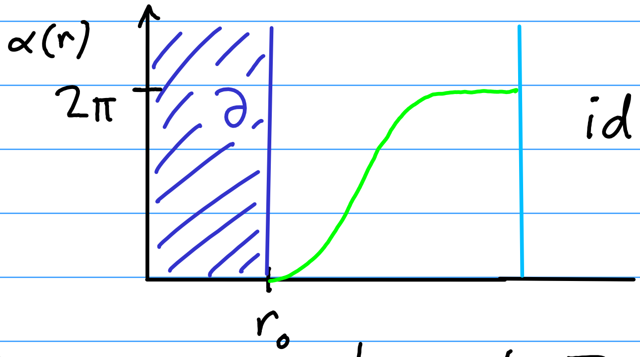
We obtain $\mathcal{M}_{g,0,b}$ from $\mathcal{M}_{g,b,0}$ by "filling up" the punctures. It's easy to see that the kernel of i_* is generated by boundary "twist maps": or Dehn twists

A Dehn twist puts two diffeos in different isotopy classes in $\mathcal{M}_{g,b,0}$, but not in $\mathcal{M}_{g,0,b}$



Near boundary:

$$(r,\theta) \longrightarrow (r, \theta + \alpha(r))$$



Hence, the kernel is $\underbrace{\mathbb{Z} \times \cdots \times \mathbb{Z}}_{b \text{ times}}$, where each $\mathbb{Z}$ corresponds to rotations around a given punctures.