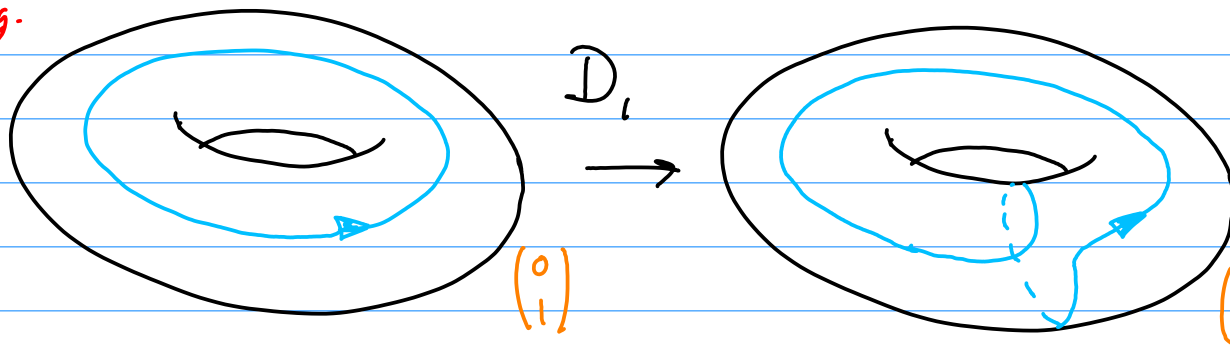
Braids Lectures 13-14: Mapping Class Group of the Torus

Consider $S_{1,0,0}$: the torus with no punctures or boundaries.

It's mapping class group $\mathcal{M}_{1,0,0}$ $= \pi_0(\mathrm{Diff}^+(T^2))$ does not involve braids, but we will use it to introduce some new concepts. The great advantage is that $\mathcal{M}_{1,0,0}$ is easy to understand.

$\mathcal{M}_{1,0,0}$ is generated by Dehn twists (or Lickorish twists):

See Farb+Marg. p. 58.



$\begin{pmatrix}0\\1\end{pmatrix} \xrightarrow{D_1} \begin{pmatrix}1\\1\end{pmatrix}$

For example, a convenient way of thinking of $\mathcal{M}_{1,0,0}$ is by looking at its action on $\pi_1(S_{1,0,0}) = \mathbb{Z}^2$:

$$D_1, D_2 \in \mathcal{M}_{1,0,0}, \quad \begin{pmatrix}p\\q\end{pmatrix} \in \pi_1(S_{1,0,0})$$

$$D_1\begin{pmatrix}p\\q\end{pmatrix} = \begin{pmatrix}p+q\\q\end{pmatrix}, \quad D_2\begin{pmatrix}p\\q\end{pmatrix} = \begin{pmatrix}p\\q-p\end{pmatrix}.$$

Here p and q just count the # of times the loop winds around each direction of the torus.

So we can write D_1 and D_2 as elements of $SL_2(\mathbb{Z})$:

$$D_1 = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \quad D_2 = \begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix},$$

$$D_1^{-1} = \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix}, \quad D_2^{-1} = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}.$$

Note that $D_1^2 D_2 D_1^2 D_2 = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$ " $D_2^2 D_1 D_2^2 D_1$ "

Hence, an arbitrary element of $\mathcal{M}_{1,0,0}$ can be written

$$[h] = \begin{pmatrix} a & b \\ c & d \end{pmatrix}, \quad a,b,c,d \in \mathbb{Z}, \quad ad - bc = 1.$$

$+1$ to preserve orientation!

Now let us explore the range of behaviour of $[h]$. Consider its characteristic polynomial

$$\det([h] - \lambda I) = \lambda^2 - \underbrace{(\operatorname{trace}[h])}_{a+d}\lambda + \underbrace{\det[h]}_{1}$$
$$= 0 \text{ for an eigenvalue.}$$

Let $\tau = \operatorname{trace}[h]$. By the Cayley-Hamilton theorem,

$$[h]^2 - \tau[h] + I = 0,$$ that is, $[h]$ satisfies its own char. poly.

The discriminant is $\tau^2 - 4$, so we have 3 possible cases:

$$|\tau| < 2, \quad |\tau| = 2, \quad |\tau| > 2.$$

Let us look at each in turn.

1. $\underline{|\tau| < 2}$: Since $[h] \in SL_2(\mathbb{Z})$, there are only a few possibilities: $\tau = 1$, $\tau = 0$, $\tau = -1$.

If $\tau = 0$, then $[h]^2 = -I$, so $[h]^4 = I$.

If $\tau = \pm 1$, then $[h]^2 \mp [h] + I = 0$

$$[h]^2 = \pm [h] - I.$$

$$\begin{aligned} [h]^3 &= \pm [h]^2 - [h] \\ &= \pm(\pm \cancel{[h]} - I) - \cancel{[h]} \\ &= \mp I \end{aligned}$$

So $[h]^6 = I$.

So, regardless of the exact value of $|\tau| < 2$, we have

$$\boxed{[h]^{12} = I} \qquad 12 = \underbrace{2 \cdot \overbrace{2 \cdot 3}^{6}}_{4}.$$

This shows that in this case $[h]$ is finite-order: we will define this more precisely later, but for now sufficient to say that $h \in [h]$ with $|\tau| < 2$ is such that h^m is isotopic to id, with $m > 0$.

2. $\underline{|\tau| = 2}$: $\quad \lambda^2 \mp 2\lambda + 1 = 0 \quad \lambda = \pm 1$ with multiplicity 2.

$\tau = \pm 2$.

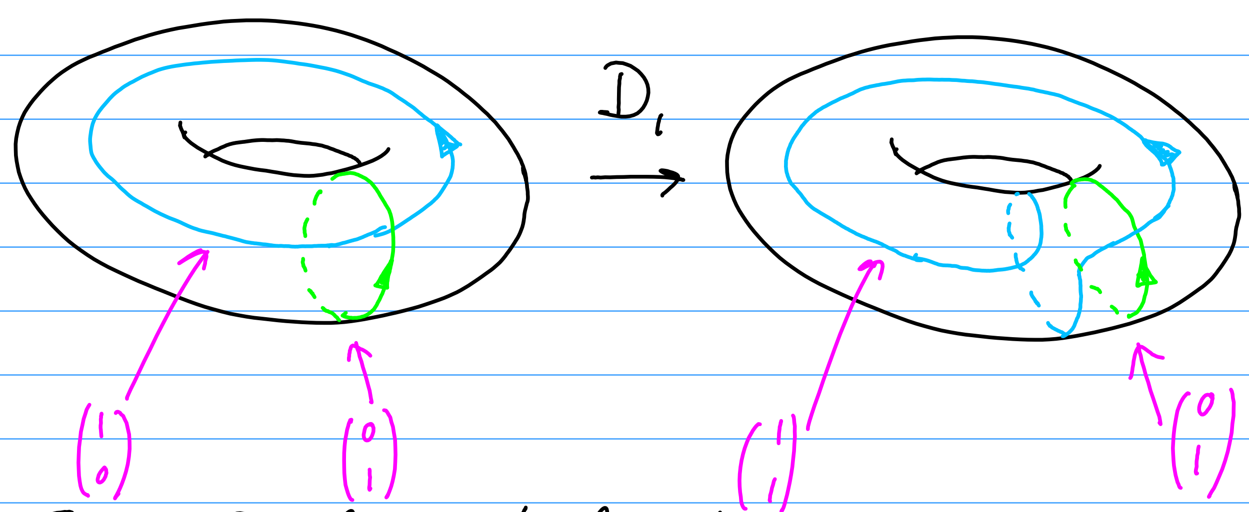
If $a = d = \pm 1$, then $[h] = \begin{pmatrix} \pm 1 & b \\ c & \pm 1 \end{pmatrix}$.

To satisfy $\det [h] = 1$, must have $bc = 0$, so either b or c or both must vanish. If both vanish then $[h]^2 = I$, so that $[h]$ is finite-order again.

Otherwise, take $b \neq 0$ and $c = 0$ WLOG.

$[h] = \begin{pmatrix} \pm 1 & b \\ 0 & \pm 1 \end{pmatrix}$, so $\begin{pmatrix} 0 \\ 1 \end{pmatrix}$ is an eigenvector with e.v. $= \pm 1$.

This eigenvector is in $\pi_1(T^2)$. It represents a loop around one of the torus' periodic directions.
Observe in particular that if we take $\lambda = +1$, then $[h] = D_1^b$, the Dehn twist.



That is: D_1 leaves the loop $\binom{0}{1}$ invariant.

Note that this does not imply that all $h \in [h]$ leaves any curve invariant, but we do know that $\exists\, h$ that does.

Now look at the other possibilities: $a+d = \pm 2$ in general (i.e., w/o $a = d = \pm 1$).

Then note that $([h] \mp I)^2 = [h]^2 \mp 2[h] + I = 0$
by Cailey-Hamilton.

$[h] \mp I$ is nilpotent, so it is $\begin{pmatrix} 0 & e \\ 0 & 0 \end{pmatrix}$ in some coordinates.

Hence, there is a change of coordinate $\underline{x}' = A\underline{x}$ such that

$$A[h]A^{-1} = \begin{pmatrix} \pm 1 & e \\ 0 & \pm 1 \end{pmatrix}$$

Jordan form

and we are back to the case $a = d = \pm 1$ above. We conclude once again that a loop (given by the sole eigenvector of $[h]$) is invariant.

$\begin{pmatrix} b \\ \pm 1 - a \end{pmatrix}$

Conclude: in all cases for $|\tau| = 2$ the action of $[h]$ on $\pi_1(T^2)$ leaves a loop invariant.

(Exception: if $[h] = \pm I$ then $[h]$ is finite-order.)

This leaves the most interesting case, $|\tau| > 2$.

3. $|\tau| > 2$: In that case we get two distinct real roots:

$$\lambda_\pm = \frac{1}{2}\left(\tau \pm \sqrt{\tau^2-4}\right), \quad \text{with } \lambda_+\lambda_- = 1$$

The roots are inverse of each other, and we define

$$\lambda = \max(|\lambda_+|, |\lambda_-|) > 1.$$

The eigenvectors of $[h]$ are $\qquad \pm = \text{sign}(\tau)$

$$u = \begin{pmatrix} \pm\lambda - d \\ c \end{pmatrix}, \quad s = \begin{pmatrix} \pm\lambda^{-1} - d \\ c \end{pmatrix}, \quad \lambda > 1$$

Check: (with $\tau > 2$)

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} \lambda - d \\ c \end{pmatrix} = \begin{pmatrix} a\lambda \underbrace{- ad + bc}_{-1} \\ c\lambda - \cancel{cd} + \cancel{cd} \end{pmatrix} = \begin{pmatrix} \underbrace{(-\lambda^2 + (a+d)\lambda - 1)}_{0} + \lambda^2 - \lambda d \\ c\lambda \end{pmatrix}$$

$$= \lambda\begin{pmatrix} \lambda - d \\ c \end{pmatrix}.$$

Claim: λ is irrational.

Assume $\tau > 2$ (positive): $\quad$ ($\Leftarrow$ $\tau < -2$ is of course almost the same proof)

$$\lambda = \frac{1}{2}\left(\tau + \sqrt{\tau^2 - 4}\right) = a_0 + \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cfrac{1}{\cdots}}}$$

$$\lambda^2 - \tau\lambda + 1 = 0 \iff \lambda - \tau + \lambda^{-1} = 0$$

$$\lambda = \underbrace{\tau - 1}_{\text{Positive integer}} + \underbrace{(1 - \lambda^{-1})}_{\substack{0 < 1 - \lambda^{-1} < 1 \\ \text{Fractional part}}} = a_0 + (1 - \lambda^{-1})$$

$$\therefore \; a_0 = \tau - 1$$

$$\lambda - a_0 = 1 - \lambda^{-1} = \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cfrac{1}{\cdots}}}$$

$$\frac{1}{1-\lambda^{-1}} = a_1 + \cfrac{1}{a_2 + \cfrac{1}{\cdots}}$$

$$1 - \lambda^{-1} = 1 - (\tau - \lambda) \quad \text{since } \lambda^2 - \tau\lambda + 1 = 0$$
$$= (1-\tau) + \lambda$$

$\underbrace{}_{}^{\phi^2!}$

Since $\lambda = \frac{1}{2}\left(\tau + \sqrt{\tau^2 - 4}\right)$ is monotonic, $\min \lambda = \frac{1}{2}\left(3 + \sqrt{5}\right)$ for $|\tau| = 3$,
$= 2.61803\ldots$

Hence, $1 < \dfrac{1}{1-\lambda^{-1}} \leq \dfrac{1}{1 - \max \lambda^{-1}}$

$\max \lambda^{-1} = (\min \lambda)^{-1} = \frac{1}{2}\left(3 - \sqrt{5}\right)$

$$= \frac{1}{\frac{1}{2}(\sqrt{5}-1)}$$
$$= \frac{2(\sqrt{5}+1)}{4} = \frac{1}{2}\left(1 + \sqrt{5}\right) < 2.$$

$\therefore 1 < \dfrac{1}{1-\lambda^{-1}} < 2.$ Conclude: $a_1 = 1.$

$$\left(\frac{1}{1-\lambda^{-1}} - 1\right)^{-1} = a_2 + \cfrac{1}{a_3 + \cfrac{1}{\cdots}}$$

$$\left(\frac{1}{1-\lambda^{-1}} - 1\right)^{-1} = \left(\frac{\cancel{1} - (\cancel{1} - \lambda^{-1})}{1 - \lambda^{-1}}\right)^{-1} = \left(\frac{1}{\lambda - 1}\right)^{-1} = \lambda - 1.$$

$$\lambda - 1 = a_2 + \cfrac{1}{a_3 + \cfrac{1}{\cdots}} \iff \lambda = \underbrace{(a_2 + 1)}_{a_0 = \tau - 1} + \cfrac{1}{a_3 + \cfrac{1}{\cdots}}$$

Same expression as when we started!

$\therefore a_2 = \tau - 2.$

One more time:

$$\lambda^2 - \tau\lambda + 1 = 0$$
$$\lambda - \tau + \lambda^{-1} = 0$$
$$\lambda - \tau + 2 = 2 - \lambda^{-1}$$

$$(\lambda - 1 - a_2)^{-1} = a_3 + \cfrac{1}{a_4 + \cfrac{1}{\cdots}}$$

$$(\lambda - 1 - a_2)^{-1} = (\lambda - \tau + 1)^{-1} = (1 - \lambda^{-1})^{-1}$$

$$\frac{1}{1 - \lambda^{-1}} = a_3 + \cfrac{1}{a_4 + \cfrac{1}{\cdots}} \qquad \therefore \quad a_3 = a_1$$
$$a_4 = a_2$$
$$a_5 = a_1$$
$$a_6 = a_2 \quad \ldots$$ Periodic!

We've seen this before!

Continued fraction representation:

$$\lambda = [\tau - 1; \underbrace{1, \tau - 2}_{\text{period-2}}, 1, \tau - 2, \ldots]$$

$\therefore$ So λ is irrational.

Of course, this shouldn't have surprised us: the irrational solutions of quadratic equations with integer coefficients are periodic. But this polynomial has a particularly simple form.

More importantly, this shows that λ is irrational for any $|\tau| > 2$ (negative τ proceeds identically)

$[h]$ is the isotopy class of an Anosov diffeomorphism with dilatation $\lambda > 1$.

More precisely, in the isotopy class $[h]$ there is a unique diffeo $h_A : T^2 \to T^2$, $T^2 = I \times I$, defined by

$$h_A(\underline{r}) = \begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} \bmod 1 \qquad \underline{r} = \begin{pmatrix} x \\ y \end{pmatrix} \in T^2 = I \times I$$

$$|a+d| > 2.$$

h_A is an Anosov diffeomorphism.

A leaf of the unstable foliation $\mathcal{F}^u$ of h_A is generated by

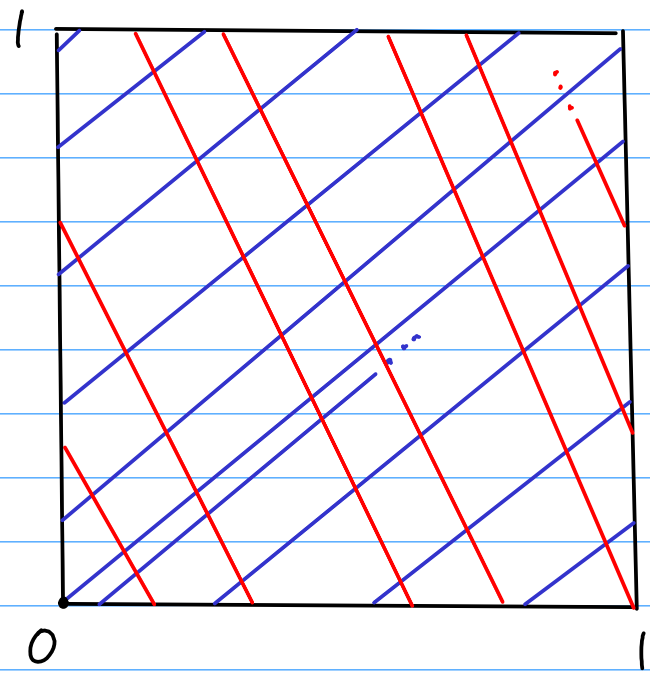
$$h_A^m(\pm u) = \lambda^m(\pm u) \bmod 1, \quad m \in \mathbb{Z}, m > 0, \quad u = \begin{pmatrix} \pm\lambda - d \\ c \end{pmatrix}$$

regarded as a curve.

Similarly, a leaf of the stable foliation $\mathcal{F}^s$ of $h_{A^{-1}}$ is generated by

$$h_A^{-m}(\pm s) = \lambda^m(\pm s) \bmod 1, \quad m \in \mathbb{Z}, m > 0, \quad s = \begin{pmatrix} \pm\lambda - d \\ c \end{pmatrix}$$

The leaves of the foliations do not carry an orientation.



Both leaves wind around the torus ∞ many times, since their slope is irrational. Each leaf is dense in T^2.

They are transverse, that is they are nowhere tangent.

The foliation $\mathcal{F}^{u,s}$ can be generated by considering the iterates of all possible curves with slope parallel to u, s.

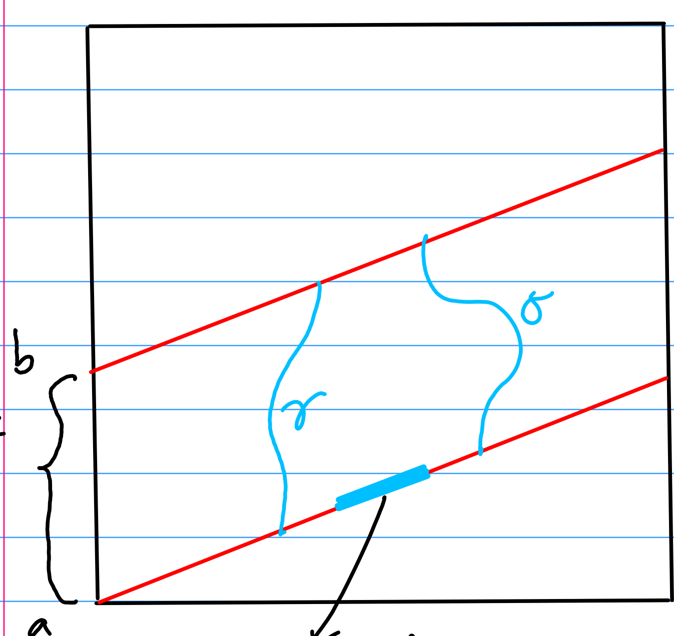
Observe that the foliations are invariant under h_A, since they have the same slope as its eigenvectors.

If we take an arbitrary curve in $\mathbb{T}^2$, then clearly it will converge to $\mathcal{F}^u$ under repeated application of h_A, or to $\mathcal{F}^s$ under repeated application of h_A^{-1}.

We can turn our foliations into measured foliations by equipping them with transverse measures μ^u and μ^s.

Obtain measured foliations $(\mathcal{F}^u, \mu^u)$ and $(\mathcal{F}^s, \mu^s)$.

The measures are the usual Borel measures, taken as proportional to the distance between leaves.



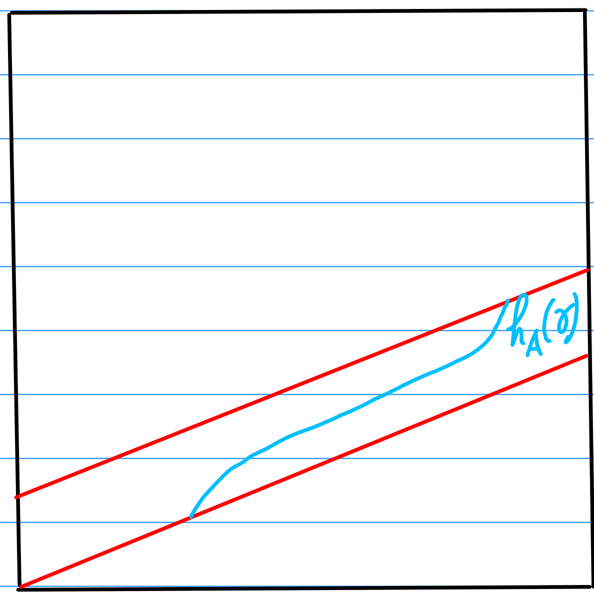
Measure of the arcs: $b-a$

This arc has zero tranverse measure.

We define the transverse measure of arcs transverse to $\mathcal{F}^u$ (not tangent to any leaf) by, for instance, the standard Borel measure as on the left. The arcs γ and σ have the same measure.

$$\mu^u(\gamma) = \mu^u(\sigma).$$

The important thing is what happens to the measure under the action of h_A.



The arc $h_A(\gamma)$ now has endpoints on different leaves, though it remains transverse. In fact,

$$\mu^u(h_A(\gamma)) = \lambda^{-1}\mu^u(\gamma)$$

If we define the action of h_A on μ^u by.

$$h_{A*}(\mu^u)(\gamma) = \mu^u(h_A(\gamma)) = \lambda^{-1}\mu^u(\gamma),$$

or

$$h_{A*}(\mu^u) = \lambda^{-1}\mu^u.$$

Similarly, we have $h_{A*}(\mu^s) = \lambda\mu^s$.

Hence, altogether we write

"Dilatation"

$$h_{A*}(\mathcal{F}^u, \mu^u) = (\mathcal{F}^u, \lambda\mu^u)$$
$$h_{A*}(\mathcal{F}^s, \mu^s) = (\mathcal{F}^s, \lambda^{-1}\mu^s)$$

This last claim requires a bit more explanation:

$$h_A(\mathcal{F}^u, \mu^u) = (\mathcal{F}^u, \lambda\mu^u)$$ and expr. for stable foliation.

why not λ^{-1}?

The new foliation is just $h_A(\mathcal{F}^u) = \mathcal{F}^u$ (invariant)

but in general it might not be!

We define the measure on $h_A(\mathcal{F}^u)$ by pulling back arcs γ transverse to $h_A(\mathcal{F}^u)$:

$$h_A^*(\mu^u)(\gamma) = \mu^u(\underbrace{h_A^{-1}(\gamma)}) = \lambda\,\mu^u(\gamma)$$

The measure of a transverse arc increases under h_A^{-1}!

So $h_A(\mathcal{F}^u, \mu^u) = (\mathcal{F}^u, \lambda\mu^u)$

Similarly, $h_A(\mathcal{F}^s, \mu^s) = (\mathcal{F}^s, \lambda^{-1}\mu^s)$,

since $h_A^*(\mu^s)(\gamma) = \mu^s(h_A^{-1}(\gamma)) = \lambda^{-1}\mu^s(\gamma)$.

Another way to look at measured foliations:

Define the foliation $\mathcal{F}^u$ in terms of the closed 1-form

$$\omega^u = dy - m^u dx$$

where m is the slope of the unstable direction:

$$m^u = \frac{c}{\lambda - d} \qquad (\tau = a+d > 2) \qquad \left(m^s = \frac{c}{\lambda^{-1} - d} \right)$$

Then we define the transverse measure $\mu^u(\gamma)$ of an arc γ by

$$\mu^u(\gamma) = \int_\gamma \omega^u = \int_\gamma dy - m^u dx$$

We can then think of leaves of $\mathcal{F}^u$ as consisting of lines of vanishing ω^u:

$$d(\underbrace{m^u x + b}_{y(x)}) - m^u dx = 0.$$

The action of h_A on μ^u can then be calculated directly:

$$\mu^u(h_A(\gamma)) = \int_{h_A(\gamma)} \omega^u(\underline{r}) = \int_{h_A(\gamma)} dy - m^u dx$$

Claim: $\omega^u(h_A(\underline{r})) = \lambda^{-1}\omega^u(\underline{r})$

$\lambda^2 - (a+d)\lambda + 1 = 0$

$\lambda = (a+d) - \lambda^{-1}$

Recall $h_A(\underline{r}) = \begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}$,

so $$\omega^u(h_A(\underline{r})) = d(cx+yd) - m^u d(ax+by)$$
$$= (c - m^u a)\,dx + (d - m^u b)\,dy$$

$$c - m^u a = c\left(1 - \frac{a}{\lambda - d}\right) = \left(\frac{\lambda - (a+d)}{\lambda - d}\right)c = -\lambda^{-1} m_u$$

$$d - m^u b = d - \frac{bc}{a - \lambda^{-1}} = \frac{d\lambda^{-1} - (ad-bc)}{a-\lambda^{-1}} = \lambda^{-1}\frac{(d-\lambda)}{a - \lambda^{-1}} = \lambda^{-1}$$

Hence, $$\omega^u(h_A(\underline{r})) = \lambda^{-1}(dy - m^u dx)$$
$$= \lambda^{-1}\omega^u(\underline{r})$$

This means that
$$\mu^u(h_A(\gamma)) = \int_{h_A(\gamma)} \omega^u(\underline{r}) = \int_\gamma \omega^u(h_A(\underline{r}))$$
$$= \lambda^{-1}\mu^u(\gamma),$$

as before.