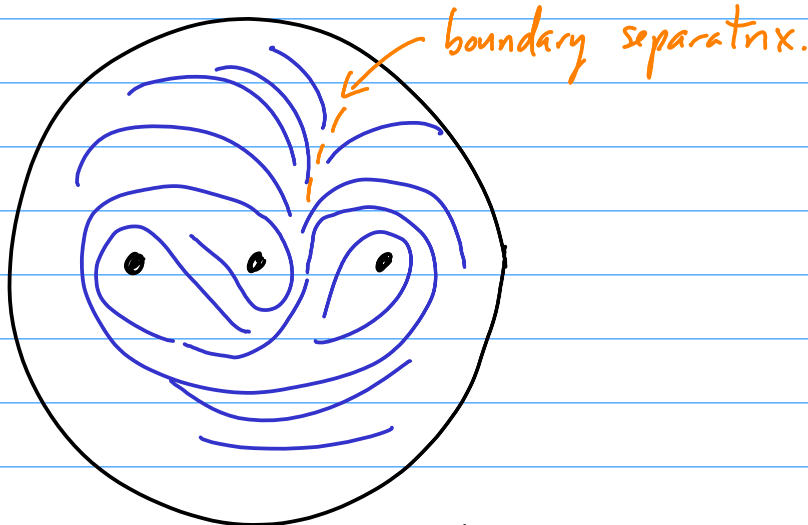
Braids Lecture 19: Singularities of Foliations

In the "slide show" we saw that as a result of stirring, a material line tends to trace out $\mathcal{F}^u$:



Consider a stirring device with n rods. The singularity data of a foliation $\mathcal{F}$ is the sequence

$$(N_{sep}, N_3, N_4, \ldots)$$

where N_{sep} is the number of separatrices (singularities) on the disc's outer boundary, and

$$N_p = \# \text{ of interior } p\text{-pronged singularities}$$

The classical Euler-Poincaré-Hopf formula for the singularities of a vector field reads

$$\sum_{i \in \operatorname{sing}(\mathcal{F})} \left(1 - \frac{p_i}{2}\right) = 2 - 2g - b$$

← "singular set of $\mathcal{F}$" ← genus ← Euler char. χ

See e.g. Milnor

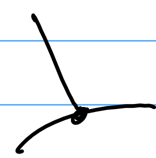
This is why the torus is so special: it can have $\mathrm{sing}(\mathcal{F}) = \phi$, since $2-2g-b=0$.

But note that for an orientable vector field $p_i =$ even. However, the formula also holds when p_i is odd. (Easy proof by "doubling the surface" at singularities.)

Here we treat the rods and outer boundary as singularities of $\mathcal{F}$. For the disc, $g=0$, $b=1$, so

$$\sum_{i \in \mathrm{sing}(\mathcal{F})} (2-p_i) = 2$$

Now, each rod has $p_i = 1$, so $2-p_i=1$.

Also, each boundary separatrix  has $p_i = 3$, $2-p_i=-1$.

Altogether, can rewrite the above in terms of the singularity data as

$$\boxed{n = N_{sep} + \sum_{p \geq 3} (p-2) N_p + 2}$$

This is the E-P-H formula adapted to stirring on a disc.
Note that $p-2 \geq 1$, so all terms on the right are positive. This severely constrains the possible # of singularity data for given n. $(N_{sep} \geq 1)$

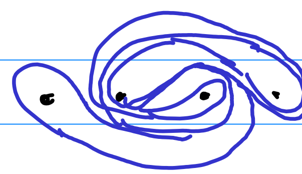
We define
$$S_n = \# \text{ of singularity data for fixed } n,$$
that is, fix n and count the possible # of choices for $(N_{sep}, N_3, N_4, \dots)$.

$n = 0, 1, 2$ are trivial (no pseudo-Anosovs)

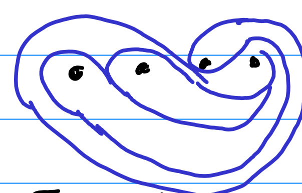
$n=3$: We need $N_{sep} \geq 1$, so $N_{sep} = 1$ and $N_p = 0$, $p \geq 3$.
$$S_1 = 1.$$

$n=4$: $$4 = N_{sep} + \sum_{p \geq 3} (p-2) N_p + 2$$
$$4 = 2 \quad 0 \leftarrow N_3 = 1 \quad 2$$
$$4 = 1 \quad 1 \quad 2$$

$\sigma_1 \sigma_2^{-1} \sigma_3 \sigma_2^{-1}$



So either $(N_{sep}, N_3, \dots) = \begin{cases} (2, 0, 0, \dots) \\ (1, 1, 0, \dots) \end{cases}$



$\sigma_1 \sigma_2 \sigma_3^{-1}$

For $n=5$, $(N_{sep}, N_3, N_4, \dots) = \begin{cases} (3, 2, 0, \dots) \\ (2, 1, 0, \dots) \\ (1, 2, 0, \dots) \\ (1, 0, 1, \dots) \end{cases}$

Every time we are "partitioning" integers.

Let $p(k)$ be the partition function, which counts the # of ways positive integers can sum to k:

$p(1) = 1$

$p(2) = 2$ since $2 = 1+1$ (and 2 itself)

$p(3) = 3$ since $3 = 1+1+1 = 1+2$

$p(4) = 5$ since $4 = 1+1+1+1 = 2+1+1 = 2+2 = 3+1$

$p(5) = 7$ By convention, $p(0) = 1$.

$p(6) = 11$

It's easy to see that

$$\boxed{S_n = \sum_{k=0}^{n-3} p(k)} \qquad n \geq 3$$

Proof (induction): We have $S_3 = 1 = p(0)$, and

$$n = N_{sep} + \sum_{p \geq 3} (p-2) N_p + 2$$

Can't go further.

$$n - 3 = \underbrace{(N_{sep} - 1)}_{N_{sep} \geq 1} + \underbrace{N_3}_{\geq 0} + 2N_4 + \ldots + (n-3) N_{n-1}$$

First define $N_{sep} = N_{sep}' + 1$, $N_{n-1} = 0$:

$$n-4 = (N_{sep}'-1) + N_3 + 2N_4 + \cdots + (n-2)N_{n-2}$$

$$(n-1)-3 = (N_{sep}'-1) + N_3 + 2N_4 + \cdots + ((n-1)-1)N_{(n-1)-1}$$

For this case, we have S_{n-1} possibilities. So there are S_{n-1} choices with just one more separatrix on the boundary.

The only cases left to consider have $N_{sep}=1$.

$$n-3 = 0 + N_3 + 2N_4 + \cdots + (n-3)N_{n-1}$$

Now if we think of N_p as the number of "blocks" of $\overbrace{1+\cdots+1}^{p-2}$

$$n-3 = \underbrace{1}_{N_3}+\underbrace{1}_{N_3}+\underbrace{1+1}_{N_4}+\underbrace{1+1}_{N_4}+\underbrace{1+1+1}_{N_5}+\cdots+1+1$$

we see the above sum "partitions" $n-3$, so the # of possibilities is $p(n-3)$. This proves that $S_n = S_{n-1} + p(n-3)$, so

$$\therefore S_n = \sum_{k=0}^{n-3} p(k)$$

Now we can ask an interesting question: how does $p(k)$ grow with large k?

The large-k asymptotics of $p(k)$ are difficult, so we state without proof the Hardy-Ramanujan formula,

$$p(k) \sim \frac{1}{4\sqrt{3}\,k} \exp\left(\pi\sqrt{\frac{2k}{3}}\right)$$

Now for large k this increases rapidly, but it is a continuous function, so we can approximate a sum by an integral over k,

$$S_n \sim \int_1^{n-3} p(k)\,dk$$

We get an "exponential integral" $Ei(x)$, which we can also approximate for large argument, and we find

$$S_n \sim \frac{1}{2\pi\sqrt{2}\,(n-3)} \exp\left(\pi\sqrt{\frac{2(n-3)}{3}}\right)$$

This agrees well with the exact formula, for large n.