Braids Lecture 20: Representations of B_n

A representation of a group G on a module M over a ring R is a group homomorphism from G to $GL(V)$.

$$\rho : G \to GL(V) \qquad \text{"left } G\text{-module"}$$

$$\rho(g_1 g_2) = \rho(g_1)\rho(g_2), \quad \forall g_1, g_2 \in G.$$

The dimension of V is the dimension of the representation. For finite-dimensional representations, we can chose a basis for V and identify $GL(V)$ with $GL(n, R)$, the group of n-by-n invertible matrices on the ring R.

The symmetric group Σ_n has a representation

Permutation matrices

$$\rho : \underbrace{\begin{pmatrix} i \\ i+1 \end{pmatrix}}_{\text{generator of } \Sigma_n} \longmapsto I_{i-1} \oplus \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \oplus I_{n-i-1}$$

$(i-1)\times(i-1)$ identity matrix. $\in GL(n, \mathbb{C})$

(It is enough to define the representation of the generators.)

The Burau representation of the braid group $B_n(\mathbb{E}^2)$ is

$$\rho : B_n(\mathbb{E}^2) \to GL(n, \mathbb{Z}[t, t^{-1}])$$

"Laurent polynomials" $t \in \mathbb{C}$, $t \neq 0$

$$\rho : \sigma_i \longmapsto I_{i-1} \oplus \begin{pmatrix} 1-t & t \\ 1 & 0 \end{pmatrix} \oplus I_{n-i-1}$$

Example: for $n=3$,

$$\rho(\sigma_1) = \begin{pmatrix} 1-t & t & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad \rho(\sigma_2) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1-t & t \\ 0 & 1 & 0 \end{pmatrix}$$

The block-structure of the Burau rep. guarantees
$\rho(\sigma_i)\rho(\sigma_j) = \rho(\sigma_j)\rho(\sigma_i)$, $|i-j|>1$, and also

$$\rho(\sigma_1)\rho(\sigma_2)\rho(\sigma_1) = \begin{pmatrix} 1-t & t(t-1) & t^2 \\ 1-t & t & 0 \\ 1 & 0 & 0 \end{pmatrix} = \rho(\sigma_2)\rho(\sigma_1)\rho(\sigma_2)$$

will be true for any 3×3 sub-block in $\rho(\sigma_i)\rho(\sigma_{i+1})\rho(\sigma_i) = \rho(\sigma_{i+1})\rho(\sigma_i)\rho(\sigma_{i+1})$.

The blocks $\begin{pmatrix} 1-t & t \\ 1 & 0 \end{pmatrix}$ reduce to the representation for Σ_n above for $t=1$. $\longrightarrow$ "deformation" of standard rep. for Σ_n.

Just like the representation for Σ_n, the Burau rep. is reducible, that is, it can be isomorphically mapped to smaller matrices, the reduced Burau representation, is

$$\rho : B_n(\mathbb{E}^2) \longrightarrow GL(n-1, \mathbb{Z}[t,t^{-1}])$$

$t \in \mathbb{C}$
$t \neq 0$

$$\rho : \sigma_i \longmapsto I_{i-2} \oplus \begin{pmatrix} 1 & -t & 0 \\ 0 & -t & 0 \\ 0 & -1 & 1 \end{pmatrix} \oplus I_{n-i-2}$$

At the (i,i) spot on diagonal

Note that we must "truncate" for $i=1$ or $i=n-1$, so that

$$\rho(\sigma_1) = \begin{pmatrix} -t & 0 \\ -1 & 1 \end{pmatrix}, \quad \rho(\sigma_2) = \begin{pmatrix} 1 & -t \\ 0 & -t \end{pmatrix}$$

$$\rho(\sigma_1)\rho(\sigma_2)\rho(\sigma_1) = \begin{pmatrix} -t & t^2 \\ -1 & 0 \end{pmatrix}\begin{pmatrix} -t & 0 \\ -1 & 1 \end{pmatrix} = \begin{pmatrix} 0 & t^2 \\ t & 0 \end{pmatrix}$$

$$\rho(\sigma_2)\rho(\sigma_1)\rho(\sigma_2) = \begin{pmatrix} 0 & -t \\ t & -t \end{pmatrix}\begin{pmatrix} 1 & -t \\ 0 & -t \end{pmatrix} = \begin{pmatrix} 0 & t^2 \\ t & 0 \end{pmatrix}$$

A standard choice is to take $t=-1$, which gives a representation $\rho : B_n \to SL(n-1, \mathbb{Z})$.

We will come back to Burau soon, but first we introduce another representation, this time infinite-dimensional.

Recall our free group on n symbols, $\mathbb{F}_n$. Write the symbols as

$$\langle x_1, \dots, x_n \rangle .$$

Then an element of $\mathbb{F}_n$ is written $x_{i_1} x_{i_2} \cdots x_{i_k}$.

We define a representation on <u>right automorphisms</u> of $\mathbb{F}_n$ ($\mathrm{Aut}\, \mathbb{F}_n$), by

- automorphisms map $\mathbb{F}_n$ to itself
- acts "from the right"

$$x_i \overline{\sigma_i} = x_i x_{i+1} x_i^{-1} \qquad \sigma_i \in B_n$$

$$x_{i+1} \overline{\sigma_i} = x_i \qquad \overline{\sigma_i} \in \operatorname{Aut} \mathbb{F}_n$$

$$x_j \overline{\sigma_i} = x_j \;, \quad j \neq i, i+1$$

Let's show that it satisfies the necessary properties:

$$\begin{aligned}
x_i \overline{\sigma_i \sigma_{i+1} \sigma_i} &= x_i x_{i+1} x_i^{-1} \overline{\sigma_{i+1} \sigma_i} \\
&= (x_i \overline{\sigma_{i+1}})(x_{i+1} \overline{\sigma_{i+1}})(x_i^{-1} \overline{\sigma_{i+1}}) \overline{\sigma_i} \\
&= (x_i x_{i+1} x_{i+2} x_{i+1}^{-1} x_i^{-1}) \overline{\sigma_i} \\
&= (x_i x_{i+1} \cancel{x_i^{-1}})(\cancel{x_i})(x_{i+2})(\cancel{x_i^{-1}})(\cancel{x_i} x_{i+1}^{-1} x_i^{-1})
\end{aligned}$$

$$\begin{aligned}
x_i \overline{\sigma_{i+1} \sigma_i \sigma_{i+1}} &= x_i \overline{\sigma_i \sigma_{i+1}} = x_i x_{i+1} x_i^{-1} \overline{\sigma_{i+1}} \\
&= (x_i)(x_{i+1} x_{i+2} x_{i+1}^{-1})(x_i^{-1})
\end{aligned}$$

Those two are equal. Next we consider

$$x_{i+1} \overline{\sigma_i \sigma_{i+1} \sigma_i} = x_i \overline{\sigma_{i+1} \sigma_i} = x_i \overline{\sigma_i} = x_i x_{i+1} x_i^{-1}$$

$$\begin{aligned}
x_{i+1} \overline{\sigma_{i+1} \sigma_i \sigma_{i+1}} &= x_{i+1} x_{i+2} x_{i+1}^{-1} \overline{\sigma_i \sigma_{i+1}} \\
&= (x_i)(x_{i+2})(x_i^{-1}) \sigma_{i+1} \\
&= x_i x_{i+1} x_i^{-1}
\end{aligned}$$

Equal again!

$$x_{i+2}\overline{\sigma_i\sigma_{i+1}\sigma_i} = x_{i+2}\overline{\sigma_{i+1}\sigma_i} = x_{i+1}\overline{\sigma_i} = x_i$$

$$x_{i+2}\overline{\sigma_{i+1}\sigma_i\sigma_{i+1}} = x_{i+1}\overline{\sigma_i\sigma_{i+1}} = x_i\overline{\sigma_{i+1}} = x_i$$

... equal yet again, and finally

$$x_j\overline{\sigma_i\sigma_{i+1}\sigma_i} = x_j\overline{\sigma_{i+1}\sigma_i\sigma_{i+1}} \quad \text{for } j \neq i, i+1.$$

So we have shown that the main braid relations are satisfied. The commutativity conditions also follow:

$$x_i\overline{\sigma_i\sigma_j} = x_i x_{i+1} x_i^{-1}\overline{\sigma_j} = x_i x_{i+1} x_i^{-1}$$

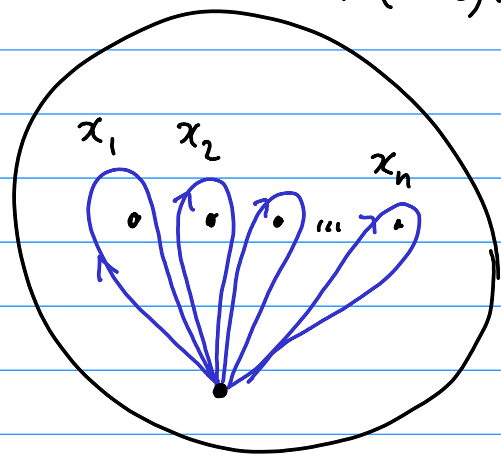
$$x_i\overline{\sigma_j\sigma_i} = x_i\overline{\sigma_i} = x_i x_{i+1} x_i^{-1} \qquad |j-i| > 1$$

$$x_{i+1}\overline{\sigma_i\sigma_j} = x_i\overline{\sigma_j} = x_i$$

$$x_{i+1}\overline{\sigma_j\sigma_i} = x_{i+1}\overline{\sigma_i} = x_i \quad , \qquad |j-i| \geq 1.$$

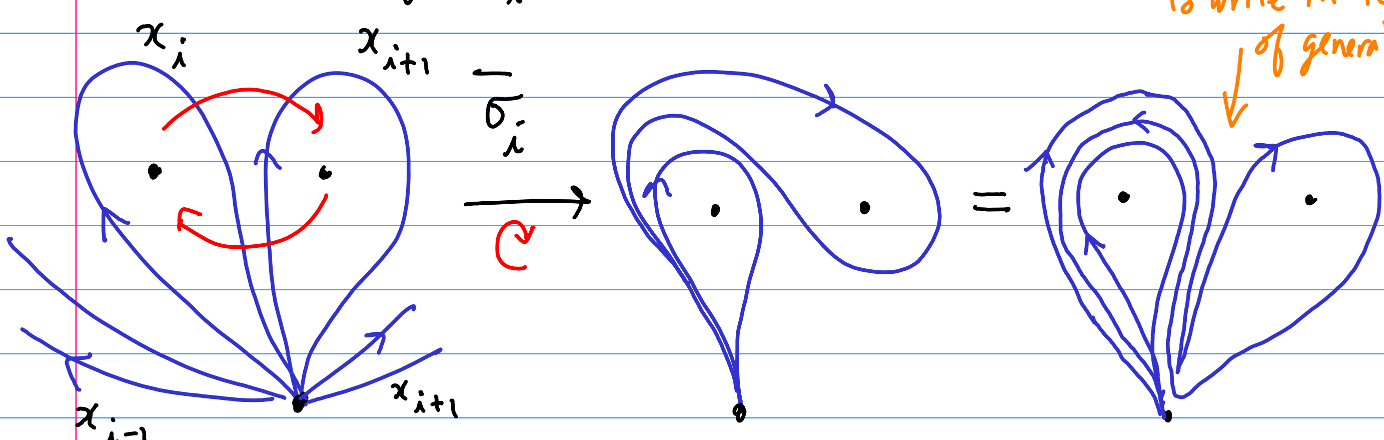
We conclude that this gives an infinite-dimensional representation of B_n as a subgroup of $\mathrm{Aut}(\mathbb{F}_n)$. Artin proved that this representation is faithful, that is, it is a group isomorphism.

The neat fact is that there is a nice geometrical interpretation for the representation. Let x_i, $i=1,\cdots,n$ be the generators for $\pi_1(S_{0,1,n})$:



Now think of $\bar{\sigma}_i$ as twisting those loops:

"Pin back down" to write in terms of generators



Hence,
$$x_i \bar{\sigma}_i = x_i x_{i+1} x_i^{-1}$$
$$x_{i+1} \bar{\sigma}_i = x_i$$
$$x_j \bar{\sigma}_i = x_j, \quad j \neq i, i+1$$

Same rules as before!