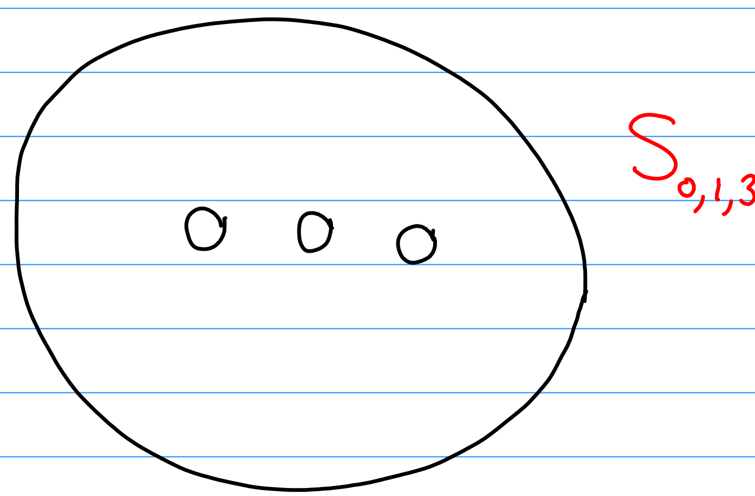
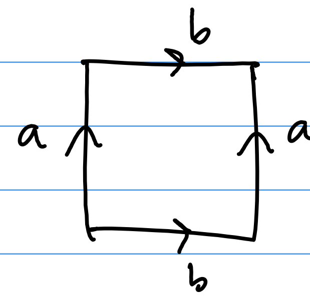
Braids Lecture 21: Burau and Homology

Last time we saw two representations of $B_n(\mathbb{E}^2)$: the Burau representation over $GL(n, \mathbb{Z}[t,t^{-1}])$ (or its reduced counterpart over $GL(n-1, \mathbb{Z}[t,t^{-1}])$), and a representation of B_n as elements of $\mathrm{Aut}(\mathbb{F}_n)$ — automorphisms of the free group. This last representation was interpreted as the action of B_n on $\pi_1(S_{0,1,n})$.

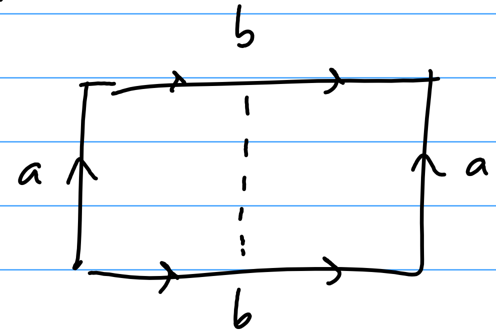
Now we will see how to derive Burau from the action on π_1. Consider again our punctured disc:



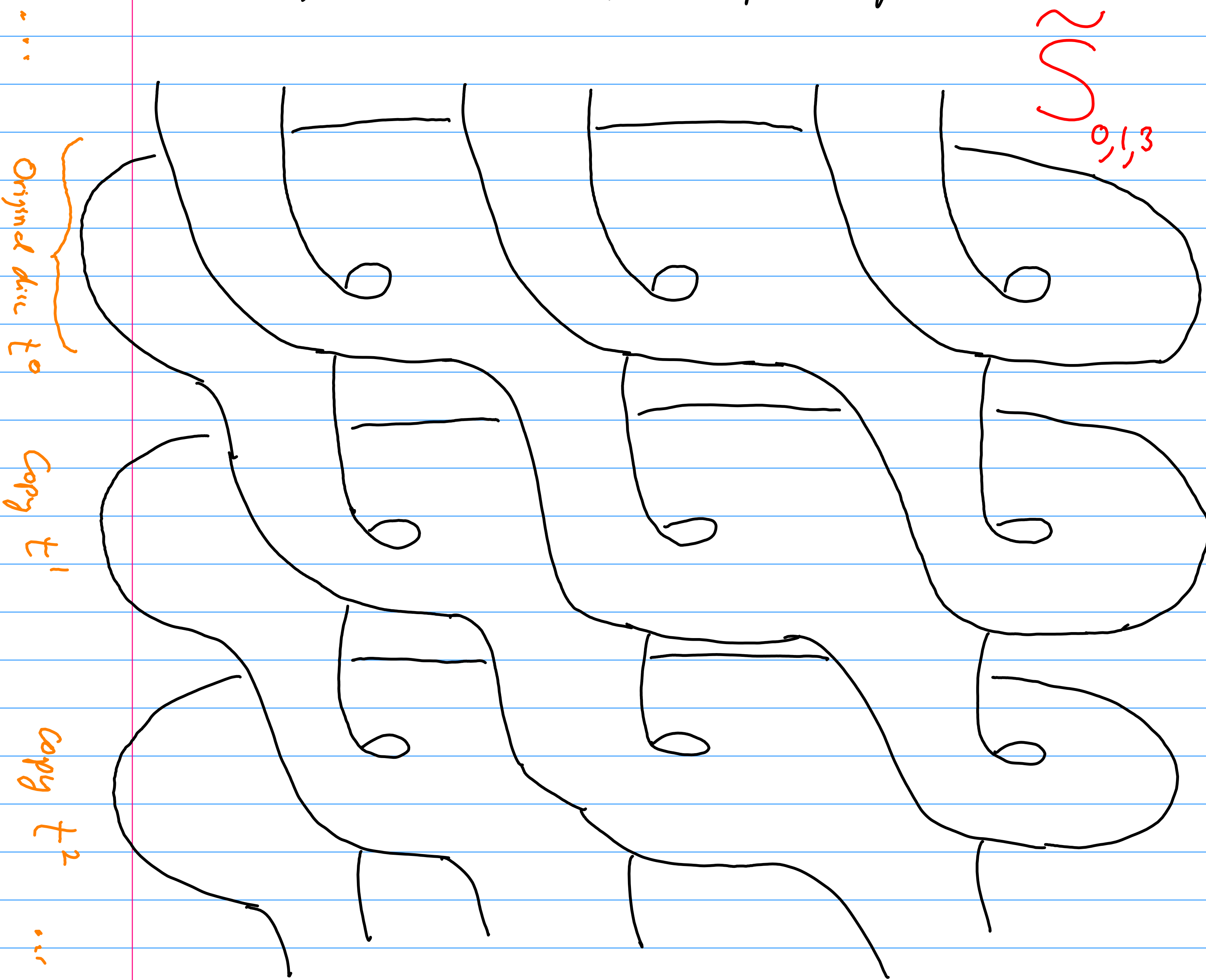
A cover (or covering space) of the disc is like a replica of the disc, many times over. For example, if we draw the torus as a square:



Then 2 squares are a double-cover of the torus

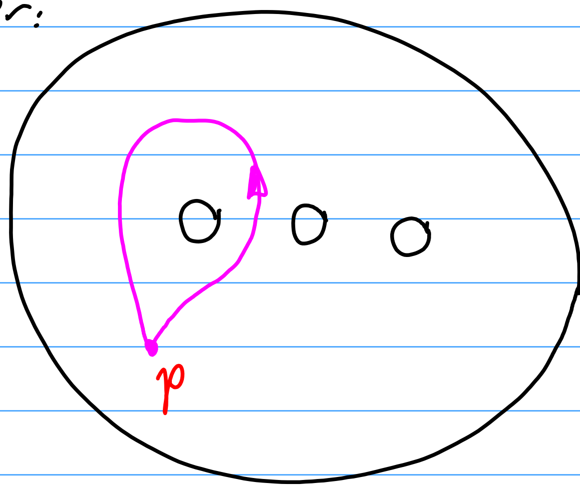


For the disc, we create a $\mathbb{Z}$-cover, meaning that each integer labels a different "part" of the cover:

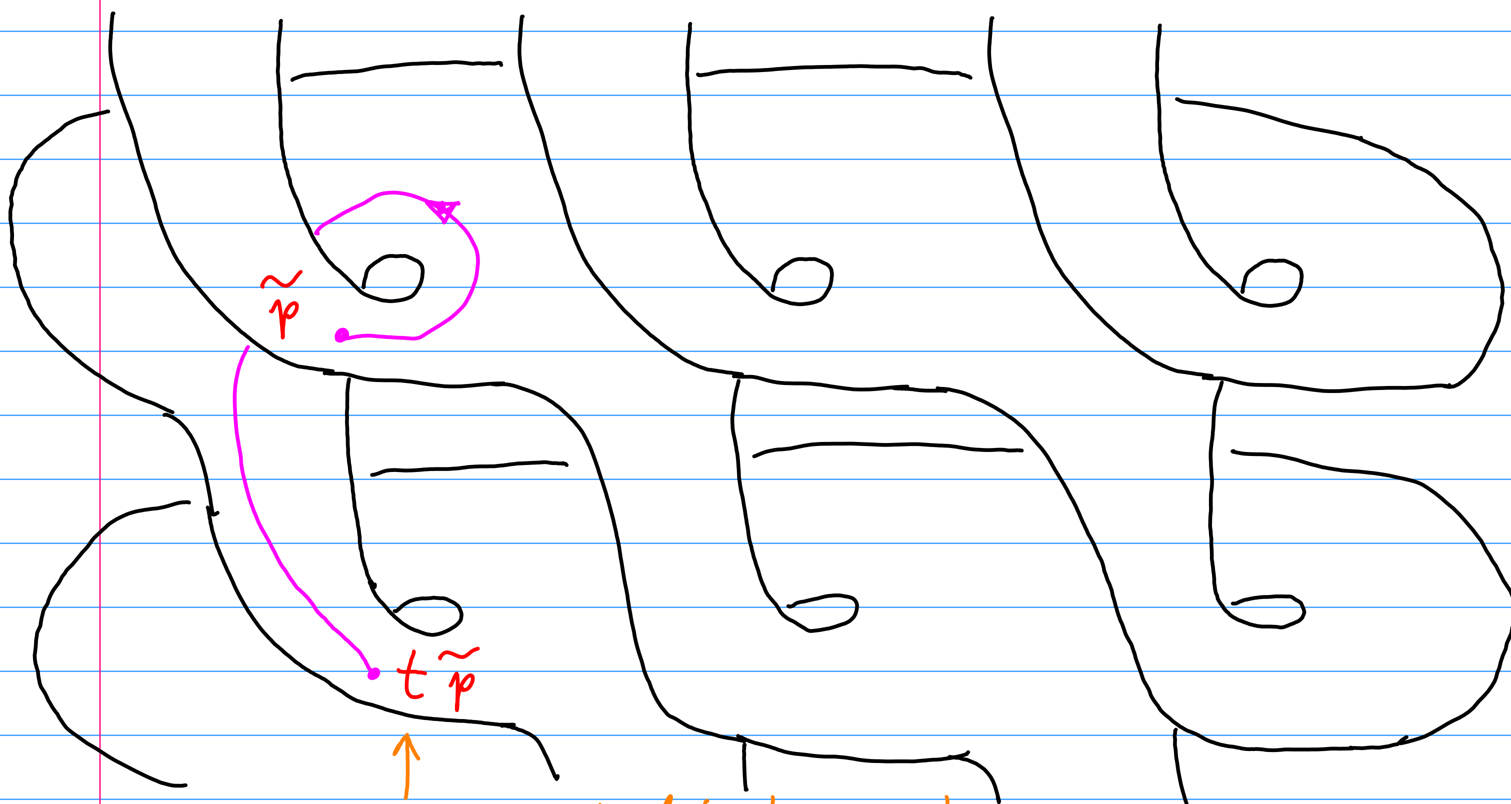


Each "copy" of the initial disc is labeled by an integer. Thus, the <u>deck transformations</u> map from one copy to another are given by the group $\langle t \rangle$, the cyclic group of infinite order.

A closed loop in $S_{0,1,0}$ is not a closed loop on the cover:

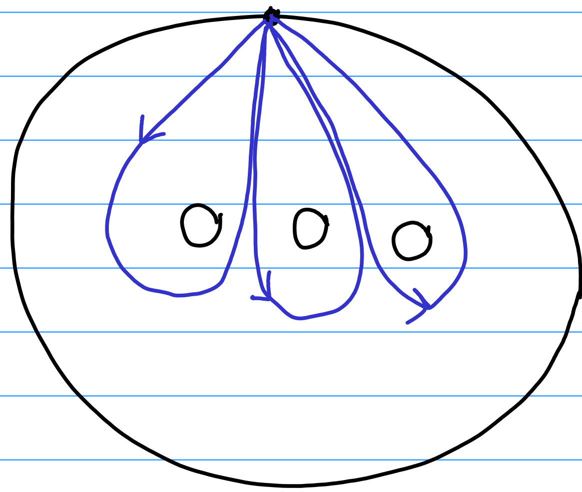


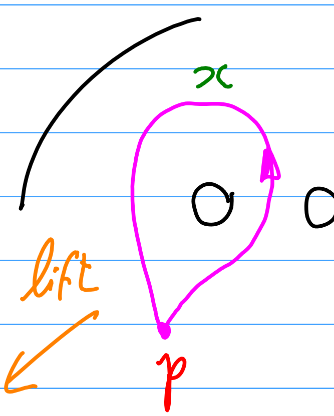
But a loop "lifts" uniquely to a path on the cover. (Recall the homotopy lifting property.)



The action of deck transformations on a point.

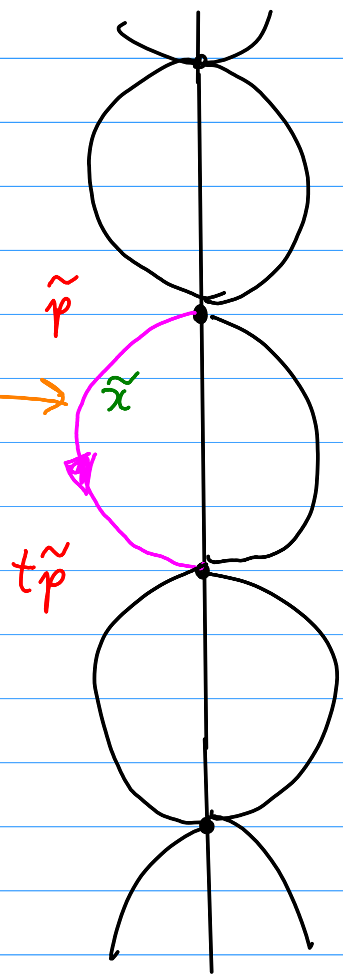
There is a simpler graphical representation of all this. Instead of taking the whole disc, take the generators of its fundamental group: ("1-skeleton of $S_{0,1,3}$")

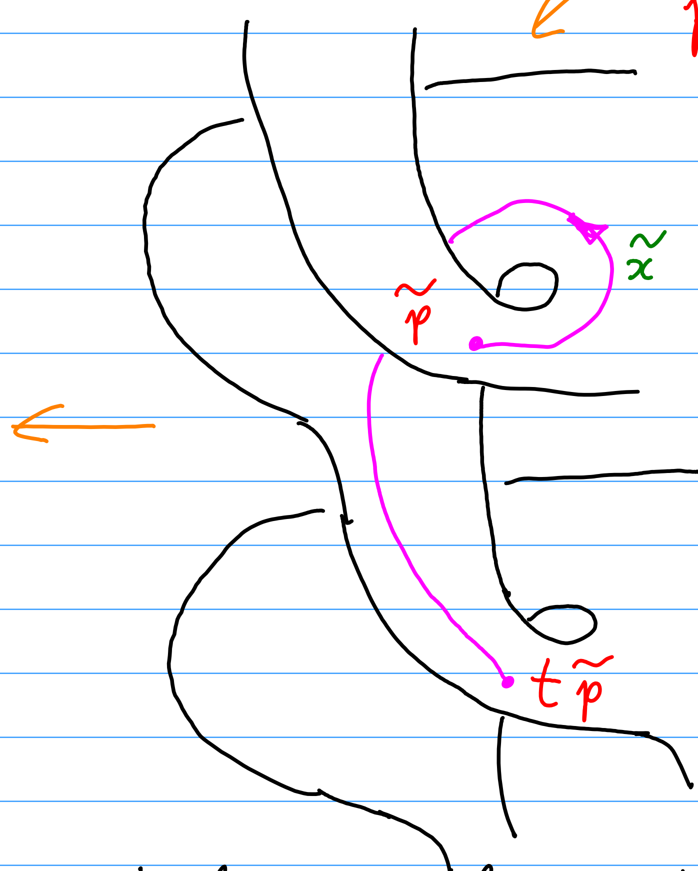




The 1-skeleton has its own $\mathbb{Z}$-cover:

This is the lift of a generator of $\pi_1(S_{0,1,3})$





In the language of homology, the loops lift to 1-simplexes in the cover.

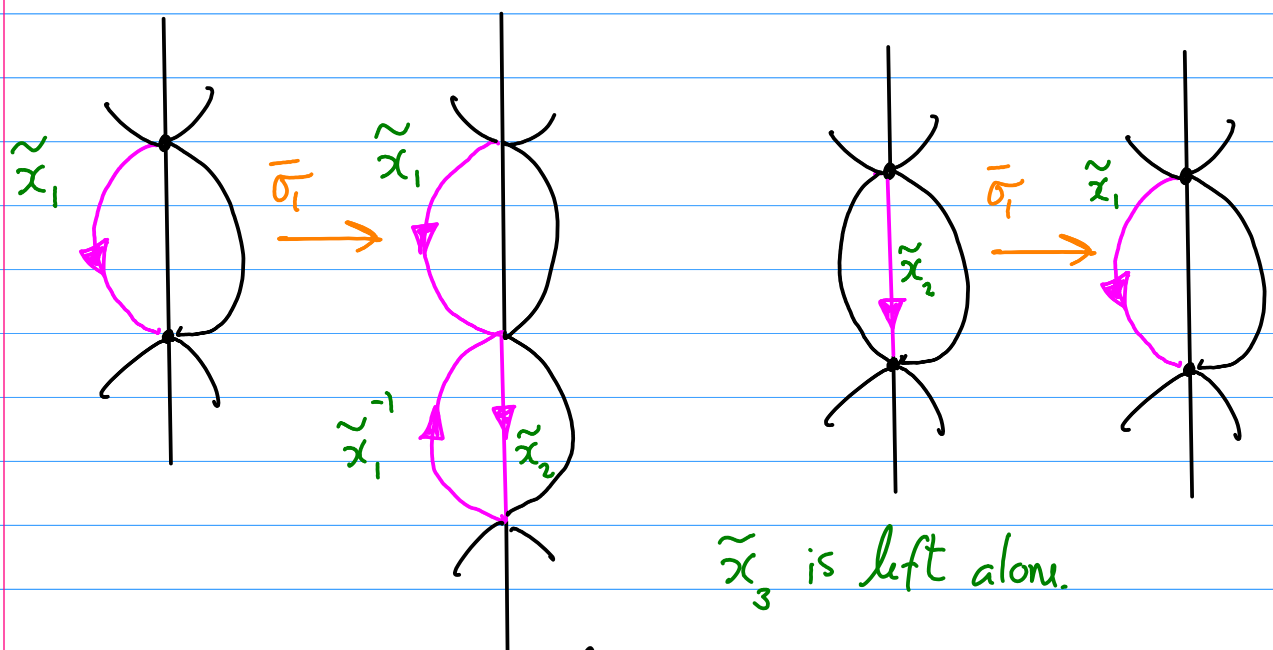
Now recall the action of σ_i on the generators of the fundamental group:

$$x_i \bar{\sigma}_i = x_i x_{i+1} x_i^{-1}$$

$$x_{i+1} \bar{\sigma}_i = x_i$$

$$x_j \bar{\sigma}_i = x_j \quad , \quad j \neq i, i+1$$

This induces an action $\bar{\sigma}_1$ on our $\mathbb{Z}$-cover:



$\tilde{x}_3$ is left alone.

In other words, over homology, $\bar{\sigma}_1$ induces an action σ_{1*}:

$$\tilde{x}_1 \longmapsto \tilde{x}_1 + t\,\tilde{x}_2 - t\,\tilde{x}_1$$

$$\tilde{x}_2 \longmapsto \tilde{x}_1$$

$$\tilde{x}_3 \longmapsto \tilde{x}_3$$

The t indicates that $\tilde{x}_1$ and $\tilde{x}_2$ live in the next copy of the 1-skeleton on the cover.

Here, the homology group is $H_1(\tilde{S}_{0,1,3}, \mathbb{Z}[t,t^{-1}])$

In terms of matrices, we can write

$$\sigma_{1x}: \begin{pmatrix} \tilde{x}_1 \\ \tilde{x}_2 \\ \tilde{x}_3 \end{pmatrix} \longmapsto \begin{pmatrix} 1-t & t & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \tilde{x}_1 \\ \tilde{x}_2 \\ \tilde{x}_3 \end{pmatrix}$$

This is precisely the Burau representation for B_3. The generalisation to B_n is obvious.

This construction — passing from homotopy to homology — is known as Abelianising. The parameter t only counts how many times we loop around a puncture: it doesn't record the order.

The reduced Burau representation is obtained by choosing the basis

$$(\tilde{x}_1 + t\tilde{x}_2 + t^2\tilde{x}_3,\ \tilde{x}_2 - \tilde{x}_1,\ \tilde{x}_2 - \tilde{x}_3).$$

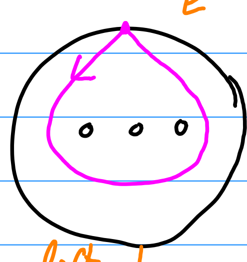
Then:

$$\tilde{x}_1 + t\tilde{x}_2 + t^2\tilde{x}_3 \longmapsto (\tilde{x}_1 + t\tilde{x}_2 - \cancel{t\tilde{x}_1}) + (\cancel{t\tilde{x}_1}) + t^2\tilde{x}_3$$
$$= \tilde{x}_1 + t\tilde{x}_2 + t^2\tilde{x}_3$$

invariant

$$\tilde{x}_2 - \tilde{x}_1 \longmapsto \cancel{\tilde{x}_1} - (\cancel{\tilde{x}_1} + t\tilde{x}_2 - t\tilde{x}_1) = -t(\tilde{x}_2 - \tilde{x}_1)$$

$$\tilde{x}_2 - \tilde{x}_3 \longmapsto \tilde{x}_1 - \tilde{x}_3 = -(\tilde{x}_2 - \tilde{x}_1) + (\tilde{x}_2 - \tilde{x}_3)$$



lifts to $\tilde{x}_1 + t\tilde{x}_2 + t^2\tilde{x}_3$

Hence, $$\sigma_{1*} : \begin{pmatrix} \tilde{x}_1 + t\tilde{x}_2 + t^2\tilde{x}_3 \\ \tilde{x}_2 - \tilde{x}_1 \\ \tilde{x}_2 - \tilde{x}_3 \end{pmatrix} \longmapsto \left(\begin{array}{c:cc} 1 & 0 & 0 \\ \hdashline 0 & -t & 0 \\ 0 & -1 & 1 \end{array}\right) \begin{pmatrix} \tilde{x}_1 + t\tilde{x}_2 + t^2\tilde{x}_3 \\ \tilde{x}_2 - \tilde{x}_1 \\ \tilde{x}_2 - \tilde{x}_3 \end{pmatrix}$$

reduced Burau

... and similarly for σ_{2*}:

$$\sigma_{2*} : \begin{pmatrix} \tilde{x}_1 + t\tilde{x}_2 + t^2\tilde{x}_3 \\ \tilde{x}_2 - \tilde{x}_1 \\ \tilde{x}_2 - \tilde{x}_3 \end{pmatrix} \longmapsto \left(\begin{array}{c:cc} 1 & 0 & 0 \\ \hdashline 0 & 1 & -t \\ 0 & 0 & -t \end{array}\right) \begin{pmatrix} \tilde{x}_1 + t\tilde{x}_2 + t^2\tilde{x}_3 \\ \tilde{x}_2 - \tilde{x}_1 \\ \tilde{x}_2 - \tilde{x}_3 \end{pmatrix}$$

Everything blocks up.