Braids Lecture 22: Topological Entropy

If $f: X \to X$ is a map, how do we measure the "complexity" of f?

One way might be to look at the number of fixed points of f^n:

$$N_n(f) = \#\text{ of fixed points of } f^n$$

Example: For a linear toral homeomorphism f,

$$\begin{pmatrix} x' \\ y' \end{pmatrix} = \underbrace{\begin{pmatrix} a & b \\ c & d \end{pmatrix}}_{M} \begin{pmatrix} x \\ y \end{pmatrix} \bmod 1, \quad \begin{array}{l} a,b,c,d \in \mathbb{Z} \\ ad-bc = 1 \\ |a+d| \neq 2 \end{array}$$

$a+d=2$ requires special care

the number of fixed points is exactly equal to

$$N_1(f) = |\operatorname{trace} M - 2|$$

[Q: Is there a simple proof of this? Doing it directly is messy.]

For $|\operatorname{trace} M| > 2$ (the Anosov case), the elements of M^n grow as λ^n for large n. So $|\operatorname{trace} M| \sim \lambda^n$, and

$$N_n(f) \sim \lambda^n.$$

The number of fixed point grows exponentially.

Equivalently, we can say that the # of periodic orbits of f (fixed points of f^n) grows exponentially with n.

This growth of periodic orbits is not, however, such a good measure of complexity. Consider the map

$$f \times R_\theta : X \times \underset{\text{1-torus}}{\mathbb{T}} \to X \times \mathbb{T}$$

defined by $(f \times R_\theta)(x, \alpha) = (f(x), \underbrace{\theta + \alpha}_{\text{rotation by } \alpha})$.

For α irrational, $(f \times R_\theta)$ has no fixed points, even though its dynamics should be "as complex" as f!

The <u>topological entropy</u> was introduced to avoid this problem.

Recall that an <u>open cover</u> $\mathcal{A}$ of a topological space X is a collection of open sets

$$\mathcal{A} = \{A_i\}_{i \in I}$$

such that $X \subseteq \bigcup_{i \in I} A_i$.

For two open covers $\mathcal{A}$ and $\mathcal{B}$, define the open cover

$$\mathcal{A} \vee \mathcal{B} = \{A_i \cap B_j\}_{\substack{i \in I \\ j \in J}}$$

A <u>subcover</u> of a cover $\mathcal{A}$ is a subset of $\mathcal{A}$ that still covers X.

Definition: Let $f: X \to X$ be a continuous map of a compact topological space X. If $\mathcal{A}$ is an open cover of X, define

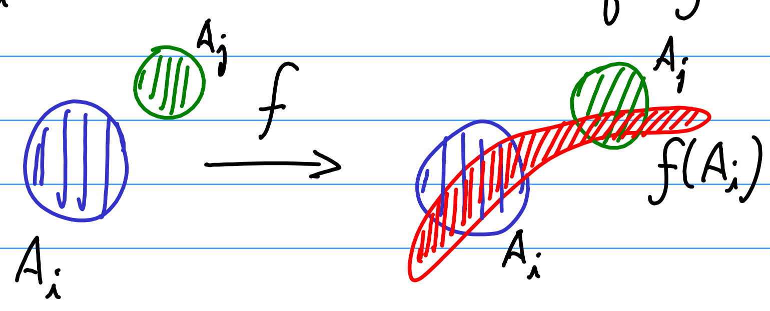
$$N_n(f, \mathcal{A}) = \text{minimum cardinality of a subcover of } \mathcal{A} \vee f^{-1}(\mathcal{A}) \vee f^{-2}(\mathcal{A}) \vee \cdots \vee f^{-n+1}(\mathcal{A})$$

$(n \geq 1)$

$$h(f, \mathcal{A}) = \limsup_{n \to \infty} \frac{1}{n} \log N_n(f, \mathcal{A}).$$

The topological entropy of f is $h(f) = \sup_{\mathcal{A}} h(f, \mathcal{A})$, where the sup is over all open covers of X.

Think of an open set in $\mathcal{A}$ as an imprecise "measurement" of a point: we know the value we want is in A_i, we're just not sure where. Now say we want to keep track of where our point in A_i could be under iteration of f.



Now our point, which was somewhere in A_i, could be in A_i or A_j. (or some other A_k's)

Taking the intersection of a, $f(a)$, $f^{-1}(a)$, $f^{-2}(a)$, ... thus creates a finer and finer cover, with a large number of components. The rate of growth of the number of components is the topological entropy (after the sup).

Thus the T.E. measures the "loss of information" about our initial measurement: initially it was in A_i, but after several iterations of f the point could be anywhere (if T.E. > 0).

Now let $f: X \to X$, $g: Y \to Y$, and $k: X \to Y$ all be continuous. Suppose that k is surjective and $f \circ k = k \circ g$:

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, "k"'] & X \arrow[d, "k"] \\ Y \arrow[r, "g"'] & Y \end{tikzcd}$$

Then $h(f) \geq h(g)$.

If k is a homeomorphism, then $h(f) = h(g)$.
So topological entropy is a topological invariant.

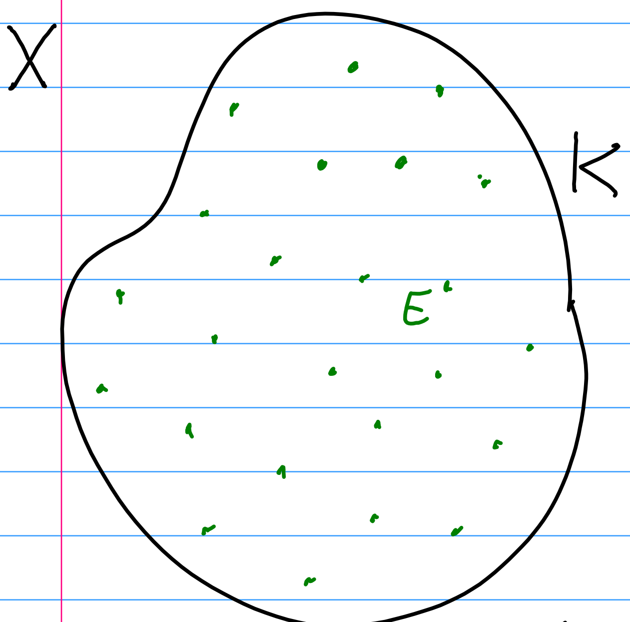
Proof: pull back the open covers of Y to open covers of X. Because k is surjective, fewer will be needed, so $h(g) \leq h(f)$.

Metric case: Bowen introduced a definition of T.E. for metric spaces.

Suppose X is a metric space (not necessarily compact) with metric $d: X \times X \to \mathbb{R}$, and $f: X \to X$ continuous. Let $K \subset X$ be compact.

$E \subset K$ is (n, ε)-separated if, given $x, y \in E$, $x \neq y$,

there is $0 \leq i < n$ s.t. $d(f^i(x), f^i(y)) \geq \varepsilon > 0$.



An (n,ε)-separated set E must consist of a finite # of isolated points.

Two points in E come at least ε apart after n iterations.

For smaller ε, can pack points more tightly. For larger n, also get more points, since they have a better chance of separating more that ε.

We let $s_K(n, \varepsilon)$ be the maximum cardinality of an (n,ε)-separated set contained in K.
We expect $s_K(n, \varepsilon)$ to increase with n and ε^{-1}.

$s_K(n,\varepsilon)$ counts the maximum number of ε-distinguishable orbits after n iterations.

The set $E \subset K$ is (n,ε)-spanning for K if for $y \in K$ there is an $x \in E$ s.t.

$$d(f^i(x), f^i(y)) < \varepsilon \;, \quad 0 \leq i < n.$$

Think of E as cell-phone towers: for every y we have a tower at x such that $f^i(y)$ stays ε-close to $f^i(x)$. We are not allowed to change towers as we go! (Also, the towers move. OK, not such a good analogy!)

We let $r_K(n,\varepsilon)$ be the minimal cardinality of an (n,ε)-spanning set contained in K.

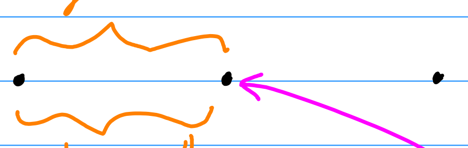
These two definitions are complementary: (n,ε)-separated requires points to get far enough from each other, (n,ε)-spanning to remain close.

But because for (n,ε)-separated the points only need to get far enough away once (for some i, $0 \leq i < n$), there are more ways to realize this, and

$$r_K(n,\varepsilon) \leq s_K(n,\varepsilon).$$

Also, the definition in (n,ε)-separated requires $x, y \in E$, whereas that in (n,ε)-spanning allows $y \in K$. So if we have an $(n, \varepsilon/2)$-separated set,

get at least $\varepsilon/2$ away from each other, but at most ε.
Otherwise we could insert another point.



stays within ε

we may or may not need this point for $s_K(n,\varepsilon)$.

Hence,

$$s_K(n,\varepsilon) \leq r_K(n,\varepsilon/2)$$

We conclude that $\quad r_K(n,\varepsilon) \leq s_K(n,\varepsilon) \leq r_K(n,\varepsilon/2)$.

Now let $\quad \bar{r}_K(\varepsilon) = \limsup\limits_{n\to\infty} \frac{1}{n} \log r_K(n,\varepsilon),$ Can show limit exists

$$\bar{s}_K(\varepsilon) = \limsup_{n\to\infty} \frac{1}{n} \log s_K(n,\varepsilon),$$

So $\quad \bar{r}_K(\varepsilon) \leq \bar{s}_K(\varepsilon) \leq \bar{r}_K(\varepsilon/2)$ Forces the limits to agree.

We define: $\quad h_K(f) = \lim\limits_{\varepsilon\to 0} \bar{s}_K(\varepsilon) = \lim\limits_{\varepsilon\to 0} \bar{r}_K(\varepsilon)$

Finally, let $h_X(f) = \sup\{h_K(f) \mid K \text{ compact in } X\}$.

When X is a compact metric space and f is continuous this agrees with our earlier definition of the topological entropy:

$$h_X(f) = h(f), \qquad X \text{ compact}, f \text{ continuous}$$

[Prove using the Lebesgue covering lemma: every open cover has a refinement consisting of ε balls]

If X is compact and $f: X \to X$ is a homeomorphism, then

$$h(f^n) = |n|\, h(f).$$