Braids Lectures 23-24: Entropy and the Fundamental Group

Having defined the topological entropy $h(f)$, we will now sketch Bowen's result that $h(f)$ is bounded below by the action of f on the fundamental group.

First, we define a measure of complexity of f by its action on any finitely generated group.

Let G be a finitely generated group with generators $\mathcal{G} = \{g_1, \dots, g_r\}$.

$L_{\mathcal{G}}(g)$ = minimum length of a word expressed in the g_i's and g_i^{-1}'s representing $g \in G$.

Let $\mathcal{G}' = \{g_1', \dots, g_s'\}$ be another set of generators.

$$L_{\mathcal{G}}(g) \leq \underbrace{\left(\max_i L_{\mathcal{G}}(g_i')\right)}_{\text{maximum length of a } g_i' \text{ expressed in terms of the } g_i\text{'s.}} L_{\mathcal{G}'}(g)$$

Let $A: G \to G$ be an endomorphism: homomorphism of G to itself, not necessarily invertible

Define:

$$\gamma_A = \sup_{g \in G} \limsup_{n \to \infty} \frac{1}{n} \log L_{\mathcal{G}}(A^n g)$$

"growth rate of A on G"

$$= \sup_{g_i \in \mathcal{G}} \limsup_{n \to \infty} \frac{1}{n} \log L_{\mathcal{G}}(A^n g_i)$$

γ_A is finite, and by the inequality above it does not depend on the set of generators:

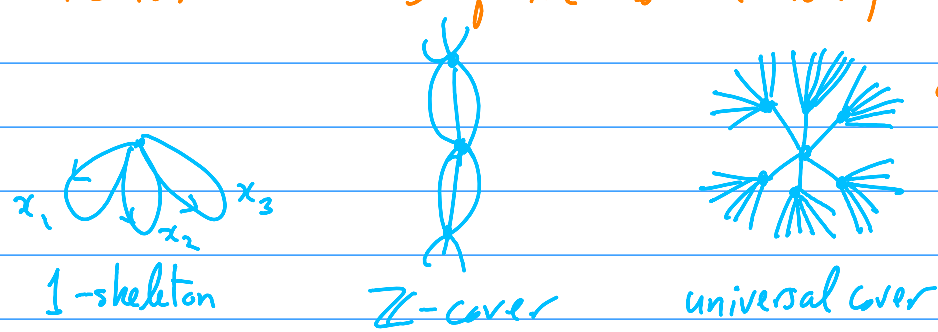
$$L_S(g) \leq \underbrace{c}_{\text{independent of } g} L_{S'}(g) \leq c\left(c' L_S(g)\right)$$

so when we take the log and the limits the constants disappear.

For G we have in mind $\pi_1(M)$, the fundamental group of a compact differentiable manifold M. But here we will think of $\pi_1(M)$ a bit differently: instead of regarding it as the space of loops, we interpret it as the group of deck transformations on the universal cover of M.

The universal cover is similar to the $\mathbb{Z}$-cover introduced last class, except that it is unique and simply-connected. If M is the punctured disc, then going around any puncture takes us to a new "copy" on the cover. Hence, the deck transformations are the loops themselves.

[In the $\mathbb{Z}$-cover from last lecture, going once in the same direction around any puncture took us to the same copy of the disc, which meant the $\mathbb{Z}$-cover wasn't simply connected. In terms of the 1-skeleton for $S_{0,1,3}$:]



← 6 branches per summit, corresponding to $x_1, x_2, x_3, x_1^{-1}, x_2^{-1}, x_3^{-1}$

Let $\tilde{M}$ denote the universal covering space of M.

If $f: M \to M$ is continuous, it lifts to $\tilde{f}: \tilde{M} \to \tilde{M}$.

If $\tilde{f}_1$ and $\tilde{f}_2$ are two liftings of f, then $\tilde{f}_1 = g\tilde{f}_2$, where g is a deck transformation.

Define $\tilde{f}_{\#}: \pi_1(M) \to \pi_1(M)$ by $\tilde{f}\alpha = \tilde{f}_{\#}(\alpha)\tilde{f}$, for $\alpha \in \pi_1(M)$ a deck transformation, $\alpha: \tilde{M} \to \tilde{M}$.

[If f is invertible, $\tilde{f}_{\#}(\alpha) = \tilde{f}\alpha\tilde{f}^{-1}$. "inner automorphisms"]

$[\tilde{f}_{\#}(\alpha)\tilde{f}_{\#}(\beta)\tilde{f} = \tilde{f}_{\#}(\alpha)\tilde{f}\beta = \tilde{f}\alpha\beta = \tilde{f}_{\#}(\alpha\beta)\tilde{f}]$ $\to$ homomorphism

For the two liftings of f above,

$$\tilde{f}_1\alpha = g\tilde{f}_2\alpha = g\tilde{f}_{2\#}(\alpha)\tilde{f}_2 = g\tilde{f}_{2\#}(\alpha)g^{-1}\tilde{f}_1 = \tilde{f}_{1\#}(\alpha)\tilde{f}_1$$

so

$$\tilde{f}_1 = g\tilde{f}_2 \implies \tilde{f}_{1\#} = g\tilde{f}_{2\#}g^{-1}$$

Now recall our definition of γ_A above: if $G = \pi_1(M)$ and the endomorphism A is $\tilde{f}_{\#}$, what is the relationship between

$$\gamma_{\tilde{f}_{1\#}} \text{ and } \gamma_{\tilde{f}_{2\#}} = \gamma_{g^{-1}\tilde{f}_{1\#}g} \text{ ?}$$

In other words, if we lift a map f to the cover, what happens to the growth rate of f on $\pi_1(M)$ if we choose a different lift?

We are rescued by:

Proposition: If $A: G \to G$ is an endomorphism and $g \in G$, and gAg^{-1} is defined by

$$[gAg^{-1}](x) = g\,A(x)\,g^{-1}, \quad x \in G,$$

then $\gamma_A = \gamma_{gAg^{-1}}$.

This is not as obvious as it might seem:

$$[gAg^{-1}]^2(x) = [gAg^{-1}]\left(gA(x)g^{-1}\right) = g\,A\left(gA(x)g^{-1}\right)g^{-1}$$

$$= g\,A(g)\,A^2(x)\,A(g^{-1})\,g^{-1}$$

$$\neq g\,A^2(x)\,g^{-1}$$

So $[gAg^{-1}]^n$ and A^n don't just differ by a constant number of g's and g^{-1}'s.

The proof is not too complicated and involves basic properties of logs of sequences, such as

$$\limsup \frac{1}{n}\log(a_n + b_n) = \max\left(\limsup \frac{1}{n}\log a_n,\ \limsup \frac{1}{n}\log b_n\right)$$

[See FLP p. 185–186 ... in English as well!]

FLP = Fathi, Laudenbach, Poénaru, "Travaux de Thurston sur les surfaces", Astérisque 66–67 (1979).

The proposition allows us to use any lift $\tilde{f}$ of f to define the action on $\pi_1(M)$ to find

$$\gamma_{f_\#} = \gamma_{\tilde{f}_\#}$$ ← we drop the ~ since doesn't depend on lift.

Remember: our goal is to relate $\gamma_{f_\#}$ and $h(f)$
[Hint: $\gamma_{f_\#} \leq h(f)$]

To do this, we will use the metric definition of $h(f)$, so we endow M with a Riemannian metric.
We can use the covering map $p: \tilde{M} \to M$ to lift the metric on M to a metric on $\tilde{M}$! The deck transformations are then isometries — they preserve the metric on $\tilde{M}$.

Crucial to our proof is the following

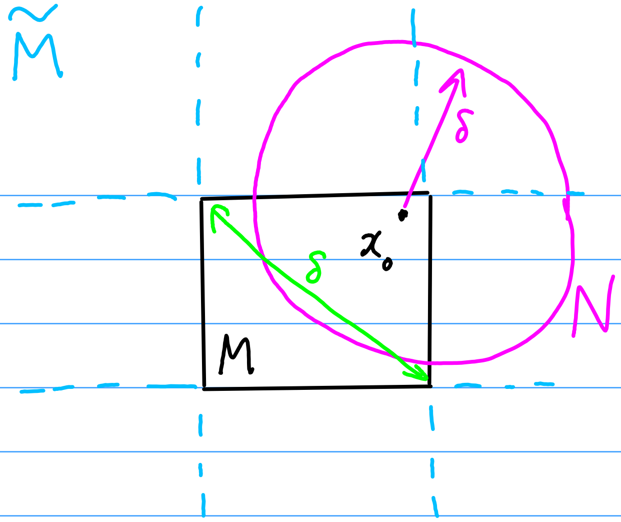
Lemma (Milnor): Fix $x_0 \in \tilde{M}$. $\exists$ constants $c_1, c_2 > 0$ such that

$$c_1 L_G(g) \leq d(x_0, gx_0) \leq c_2 L_G(g), \quad \forall g \in \pi_1(M),$$

where G is a set of generators for $\pi_1(M)$.

Proof: Let $\delta = \operatorname{diam} M \equiv \sup\{d(x,y) \mid x, y \in M\}$.

Define $N \subset \tilde{M}$ by $N = \{x \in \tilde{M} \mid d(x, x_0) \leq \delta\}$.



Since N can completely cover M after being translated, we have
$$p(N) = M.$$

For $g \in \pi_1(M)$ a deck transformation,

$\{gN\}_{g \in \pi_1(M)}$ is a (locally finite) covering of $\tilde{M}$ by compact sets. Choose as a finite set of generators
$$\underbrace{\mathcal{G} = \{g \in \pi_1(M) \mid gN \cap N \neq \phi\}}_{\text{This ensures that } \mathcal{G} \text{ is finite — } g \text{ doesn't move } N \text{ too far.}}$$

Why didn't we just any old set of generators for $\pi_1(M)$? Because now we can bound the distance travelled as we apply the generators in $\mathcal{G}$.

Now suppose $g = \underbrace{g_1 \cdot \cdots \cdot g_n}_{\text{with } g_i N \cap N \neq \phi.}$ for $g \in \pi_1(M)$. Then $L_{\mathcal{G}}(g) = n$.

At the very worst, a g_i can move us a distance 2δ on $\tilde{M}$. Hence,
$$d(x_0, gx_0) \leq 2\delta n.$$

Kind of obvious, really!

Conclude: $d(x_0, gx_0) \leq \underbrace{2\delta}_{C_2} L_{\mathcal{G}}(g)$.

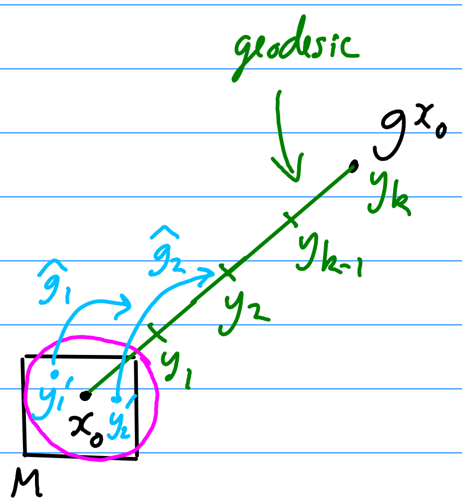
Now we need to show the lower bound.

note, unlike before

Put $\nu = \min\{d(N, gN) \mid N \cap gN = \emptyset\}$.

So ν is the least distance moved by N under $\pi_1(M)$ before it stops overlapping with itself. By compactness, $\nu > 0$: it cannot be infinitesimally close to itself.

Let k be the minimal integer s.t. $d(x_0, gx_0) < k\nu$.



Along the minimizing geodesic from x_0 to gx_0, take $k+1$ points $y_0 = x_0, y_1, \dots, y_k = gx_0$ such that

$$d(y_i, y_{i+1}) < \nu, \quad i = 0, \dots, k-1$$

Then, for $1 \leq i < k-1$, choose $y_i' \in N$ and $\hat{g}_i \in \pi_1(M)$ s.t. $y_i = \hat{g}_i y_i'$, and put $\hat{g}_0 = e$, $\hat{g}_k = g$.

We have $d(\hat{g}_i y_i', \hat{g}_{i+1} y_{i+1}')$ isometry property

$$= d(\underbrace{y_i'}_{\in N}, \hat{g}_i^{-1} \hat{g}_{i+1} \underbrace{y_{i+1}'}_{\in N}) = d(y_i, y_{i+1}) < \nu.$$

Hence, since $\hat{g}_i^{-1} \hat{g}_{i+1}$ moves a point in N such that it is still within distance $< \nu$ of N, $\hat{g}_i^{-1} \hat{g}_{i+1} N \cap N \neq \emptyset$, since ν is the least nonoverlapping distance.

But this means that $\hat{g}_i^{-1} \hat{g}_{i+1} \in G$, our previously-defined set of generators that have an overlap of N with gN.

We can then write $g = \underbrace{(\hat{g}_0^{-1}\hat{g}_1)(\hat{g}_1^{-1}\hat{g}_2)\cdots(\hat{g}_{k-1}^{-1}\hat{g}_k)}_{k \text{ factors}}$,
and

$$L_S(g) \leq k \quad \text{(there might be cancellations — recall that } L_S(g) \text{ is defined as the min over } S.)$$

Since k is the minimal integer s.t. $\frac{1}{\nu} d(x_0, gx_0) < k$, we have $\frac{1}{\nu} d(x_0, gx_0) + 1 \geq k$, and

$$L_S(g) \leq \frac{1}{\nu} d(x_0, gx_0) + 1 \leq \underbrace{\left(\frac{1}{\nu} + \frac{1}{\mu}\right)}_{c_1^{-1}} d(x_0, gx_0)$$

where $\mu = \min\{d(x_0, gx_0) \mid g \neq e,\ g \in \pi_1(M)\}$.

This completes the proof!

Now follows the easy part: consider $f: M \to M$ and $\tilde{f}: \tilde{M} \to \tilde{M}$ a lifting of f. With the lemma, we obtain

$$\gamma_{f\#} = \max_{g \in \pi_1(M)} \limsup_{n \to \infty} \frac{1}{n} \log L_S(\tilde{f}_\#^n(g))$$

$$\boxed{\gamma_{f\#} = \max_{g \in \pi_1(M)} \limsup_{n \to \infty} \frac{1}{n} \log d(x_0, \tilde{f}_\#^n(g) x_0)}$$

since c_1 and c_2 drop out.

We're getting much closer! Now at least we can relate $\gamma_{f\#}$ to the metric.

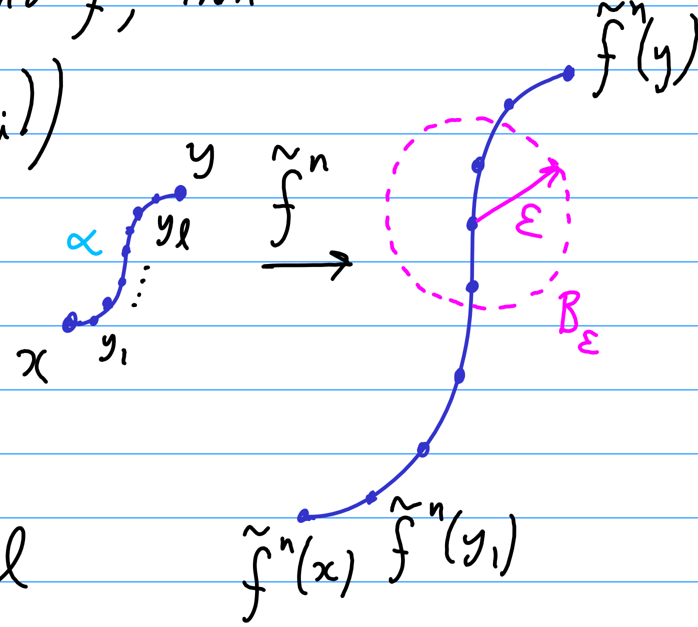
We need one more Lemma before we get to the theorem:

<u>Lemma 2:</u> Given $x, y \in \tilde{M}$,

$$\limsup_{n\to\infty} \frac{1}{n} \log d(\tilde{f}^n(x), \tilde{f}^n(y)) \leq h_M(f).$$

<u>Proof:</u> Choose an arc α from x to y. If $y_1, \dots, y_\ell$ is $(n+1, \varepsilon)$-spanning for α and $\tilde{f}$, then

$$\tilde{f}^n(\alpha) \subset \bigcup_{i=1}^{\ell} B_\varepsilon(\tilde{f}^n(y_i))$$



Since $\tilde{f}^n(\alpha)$ is connected,

$\operatorname{diam} \tilde{f}^n(\alpha) \leq 2\varepsilon\ell$,

hence $d(\tilde{f}^n(x), \tilde{f}^n(y)) \leq 2\varepsilon\ell$

Take ℓ to be minimal:

$$d(\tilde{f}^n(x), \tilde{f}^n(y)) \leq 2\varepsilon \underbrace{r_\alpha(n+1, \varepsilon)}_{\text{min. cardinality of an } (n+1,\varepsilon)\text{-spanning set for } \alpha}$$

$$\limsup_{n\to\infty} \frac{1}{n} \log d(\tilde{f}^n(x), \tilde{f}^n(y)) \leq \limsup_{n\to\infty} \frac{1}{n} \log (2\varepsilon r_\alpha(n+1, \varepsilon)) = \bar{r}_\alpha(\varepsilon)$$

Recall that $h_\alpha(\tilde{f}) = \sup_{\gamma\in\alpha} \lim_{\varepsilon\to 0} \{ \bar{r}_\gamma(\varepsilon) \mid \gamma \in \alpha \}$

Hence, $\bar{r}_\alpha(\varepsilon) \leq h_\alpha(\tilde{f})$.

But also, $h_{\tilde{M}}(\tilde{f}) = \sup_{K\subset\tilde{M}} \{ h_K(\tilde{f}) \mid K \text{ compact in } \tilde{M} \}$,

so $h_\alpha(\tilde{f}) \leq h_{\tilde{M}}(\tilde{f})$, since α is compact in $\tilde{M}$.

Finally, the fact that $h_M(f) = h_{\tilde{M}}(\tilde{f})$ follows from

Proposition 2: Suppose $p: X \to Y$ is a metric covering and $f: X\to X$, $g: Y \to Y$ are uniformly continuous. If $pg = fp$, then $h_X(f) = h_Y(g)$.

[See FLP p. 184 for a proof.] Putting $X = \tilde{M}$, $Y = M$, $f = \tilde{f}$, $g = f$ thus gives $h_M(f) = h_{\tilde{M}}(\tilde{f})$, and Lemma 2 is proved.

And we're there:

Theorem: If $f: M \to M$ is a continuous map, then
(Bowen)

$$\gamma_{f\#} \leq h_M(f).$$

Proof: $\gamma_{f\#} = \max_{g\in\pi_1(M)} \left[\limsup_{n\to\infty} \frac{1}{n} \log d(x_0, \tilde{f}^n_\#(g) x_0) \right]$,

so we have to prove that $\forall g \in \pi_1(M)$,

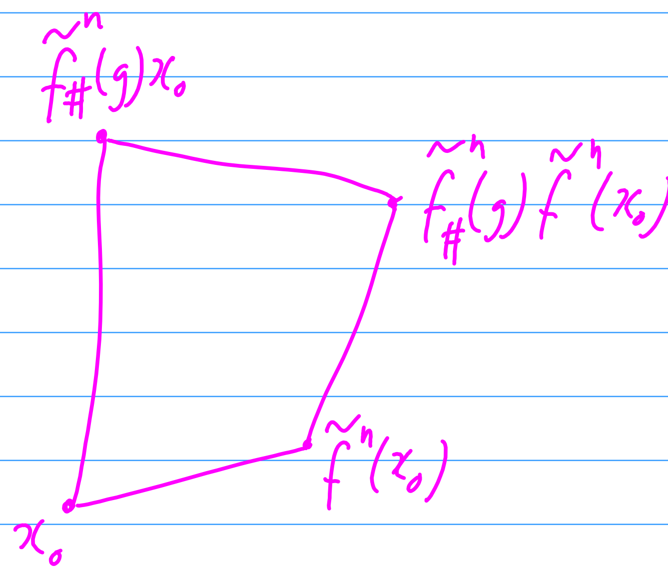
$$\limsup_{n\to\infty} \frac{1}{n} \log d(x_0, \tilde{f}^n_{\#}(g)x_0) \leq h_M(f).$$

$$d(x_0, \tilde{f}^n_{\#}(g)x_0) \leq d(x_0, \tilde{f}^n(x_0)) + d(\tilde{f}^n(x_0), \tilde{f}^n_{\#}(g)\tilde{f}^n(x_0))$$
$$+ d(\tilde{f}^n_{\#}(g)\tilde{f}^n(x_0), \tilde{f}^n_{\#}(g)x_0)$$

Recall that

$$\tilde{f}^n_{\#}(g)\tilde{f}^n = \tilde{f}^n g,$$

so $\tilde{f}^n_{\#}(g)\tilde{f}^n(x_0) = \tilde{f}^n(gx_0)$.



Also, deck transformations are isometries, so

$$d(\tilde{f}^n_{\#}(g)\tilde{f}^n(x_0), \tilde{f}^n_{\#}(g)x_0)$$
$$= d(\tilde{f}^n(x_0), x_0).$$ Hence,

$$d(x_0, \tilde{f}^n_{\#}(g)x_0) \leq 2d(x_0, \tilde{f}^n(x_0)) + d(f^n(x_0), f^n(gx_0)).$$

Now,

$$d(x_0, \tilde{f}^n(x_0)) \leq d(x_0, \tilde{f}(x_0)) + d(\tilde{f}(x_0), \tilde{f}^2(x_0)) + \cdots$$
$$+ d(\tilde{f}^{n-1}(x_0), \tilde{f}^n(x_0)).$$

$$a_n = d(\tilde{f}^{n-1}(x_0), \tilde{f}^n(x_0))$$
$$b_n = d(\tilde{f}^n(x_0), f^n(gx_0)).$$

$$d(x_0, \tilde{f}^n(x_0)) \leq 2(a_1 + \cdots + a_n) + b_n$$

drop

$$\limsup_{n\to\infty} \frac{1}{n} d(x_0, \tilde{f}^n(x_0)) \leq \max\left[\limsup \frac{1}{n} \log 2(a_1 + \cdots + a_n), \limsup \frac{1}{n} \log b_n\right]$$

Also, $\limsup \frac{1}{n} \log a_n \leq \limsup \frac{1}{n} \log (a_1 + \cdots + a_n)$

$$\leq \max\left[0, \limsup \frac{1}{n} \log a_n\right]$$

[See FLP p. 185-186]

By Lemma 2: $\limsup\limits_{n\to\infty} \frac{1}{n} \log d(\tilde{f}^n(x), \tilde{f}^n(y)) \leq h_M(f)$.

So $\limsup \frac{1}{n} \log a_n \leq h_M(f)$, $\limsup \frac{1}{n} \log b_n \leq h_M(f)$.

Hence, $\limsup\limits_{n\to\infty} \frac{1}{n} d(x_0, \tilde{f}^n(x_0)) \leq h_M(f)$,

which proves the theorem.