Braids Lecture 25: Action on $\pi_1(M)$ for the torus
Manning's Theorem

Recall from last time:

Compact metric space M, universal cover $\tilde{M}$.

Deck transformations $G = \pi_1(M)$

$f : M \to M$ continuous

$\tilde{f} : \tilde{M} \to \tilde{M}$ any lift

$\tilde{f}_{\#} : \pi_1(M) \to \pi_1(M)$ endomorphism defined by

$$\tilde{f}_{\#}(g)\,\tilde{f}(x) = \tilde{f}(gx), \qquad g \in \pi_1(M),\ x \in \tilde{M}.$$

$$\gamma_{f\#} = \text{growth rate of } \tilde{f}_{\#} \text{ acting on } \pi_1(M)$$

$$= \max_{g \in \pi_1(M)} \limsup_{n\to\infty} \frac{1}{n} \log d(x_0, \tilde{f}_{\#}^{\,n}(g)x_0), \qquad x_0 \in \tilde{M}.$$

Let's see what this all means when $M = T^2$ (torus),

$$f : M \to M \qquad f(\underline{x}) = \begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} \bmod 1, \qquad ad - bc = 1,\ a,b,c,d \in \mathbb{Z}$$

$\underline{x} = \begin{pmatrix} x \\ y \end{pmatrix}$

Then $\tilde{f}(\underline{x}) = \overbrace{\begin{pmatrix} a & b \\ c & d \end{pmatrix}}^{M}\begin{pmatrix} x \\ y \end{pmatrix}$ (no mod) $\begin{pmatrix} x \\ y \end{pmatrix} \in \tilde{M}$.
$= \mathbb{R}^2$

$$\tilde{f}^{-1}(\underline{x}) = \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = M^{-1}\underline{x}$$

Deck transformations: $g\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x+r \\ y+s \end{pmatrix}$ $r, s \in \mathbb{Z}$.

Hence, $\tilde{f}_{\#}(g)\underline{x} = \tilde{f}(g\tilde{f}^{-1}(\underline{x})) = M\left(M^{-1}\underline{x} + \begin{pmatrix} r \\ s \end{pmatrix}\right)$

$$= \underline{x} + \begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} r \\ s \end{pmatrix}$$

$$= \underline{x} + M\begin{pmatrix} r \\ s \end{pmatrix}$$

$d(\underline{x}_0, \tilde{f}^n_{\#}(g)\underline{x}_0) = \left\| M^n\begin{pmatrix} r \\ s \end{pmatrix} \right\|$ $\leftarrow$ independent of $\underline{x}_0$!

As $n \to \infty$, the largest eigenvalue of M dominates.

Hence, $\gamma_{f_{\#}} = \lambda$ for $|a+d| > 2$ (Anosov)

λ = spectral radius of M.

In fact in this case λ is the topological entropy (and not just a lower bound).

We can easily recover Manning's result for the action on homology from

Lemma: If G_1 and G_2 are finitely-generated groups, $A: G_1 \to G_1$, $B: G_2 \to G_2$, $p: G_1 \to G_2$ homomorphisms with p surjective and $pA = Bp$:

$$\begin{tikzcd} G_1 \arrow[r, "p"'] \arrow[d, "A"'] & G_2 \arrow[r] \arrow[d, "B"] & 0 \\ G_1 \arrow[r, "p"'] & G_2 \arrow[r] & 0 \end{tikzcd}$$

then $\gamma_A \geq \gamma_B$.

Apply this to:

$$\begin{tikzcd} \pi_1(M) \arrow[r, "\text{abelianize}"'] \arrow[d, "f_\#"'] & H_1(M) \arrow[r] \arrow[d, "f_{1*}"] & 0 \\ \pi_1(M) \arrow[r] & H_1(M) \arrow[r] & 0 \end{tikzcd}$$

$\uparrow$ Bowen $\qquad\qquad$ $\uparrow$ Manning