Braids Lectures 27-29: Entropy of pA's

Recall our estimate $\gamma_{f\#} \leq h(f)$.

There is a related estimate when considering $[\alpha]$, the class of loops freely homotopic to $\alpha \in \pi_1(M)$.
Let $\ell([\alpha])$ be the minimum length of a smooth loop in this class. For $f: M \to M$ continuous, define $f[\alpha]$ as a free homotopy class of loops.

Let
$$G_f([\alpha]) = \limsup_{n\to\infty} \frac{1}{n} \log\left(\ell(f^n[\alpha])\right)$$
$$G_f = \sup_\alpha G_f([\alpha]).$$

Then we can see that $G_f \leq \gamma_{f\#}$.

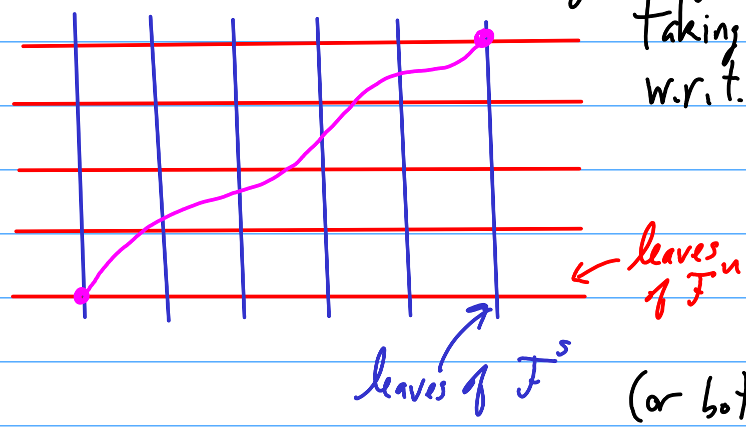
Now earlier we defined a pA diffeomorphism f as possessing invariant transverse measured singular foliations $(\mathcal{F}^u, \mu^u)$ and $(\mathcal{F}^s, \mu^s)$, such that

$$f(\mathcal{F}^u, \mu^u) = (\mathcal{F}^u, \lambda\mu^u), \quad f(\mathcal{F}^s, \mu^s) = (\mathcal{F}^s, \lambda^{-1}\mu^s).$$

Proposition: $G_f(\alpha) = \log \lambda$, for any nontrivial simple closed curve α.

The proof relies on defining a "metric" $\mu = \sqrt{(\mu^u)^2 + (\mu^s)^2}$ to measure the length of the curve.

In other words, the length of a curve is measured by taking the measure of the arc w.r.t. μ^u and μ^s.



Since the foliations are transverse, an arc of non-zero length must have a nonzero measure μ^u or μ^s (or both), so the metric is nondegenerate.

After that we use the exponential growth/decay of the measures.

[See FLP p. 178 for a proof.]

We shall need $\log \lambda = G_f$ to prove

Proposition: If $f: M \to M$ is pseudo-Anosov, then

$$h(f) = \gamma_{f\#} = \log \lambda .$$

Proof: $G_f = \log \lambda$, so we will show

$$\log \lambda = G_f \leq \gamma_{f\#} \leq \underbrace{h(f) \leq \log \lambda}_{\text{This is what we have to show.}}$$

To show this, we will find a subshift of finite type $\sigma_A: \Sigma_A \to \Sigma_A$ and a surjective continuous map $\theta: \Sigma_A \to M$ such that

$$\begin{tikzcd} \Sigma_A \arrow[r, "\sigma_A"] \arrow[d, "\theta"'] & \Sigma_A \arrow[d, "\theta"] \\ M \arrow[r, "f"] & M \end{tikzcd}$$

commutes,

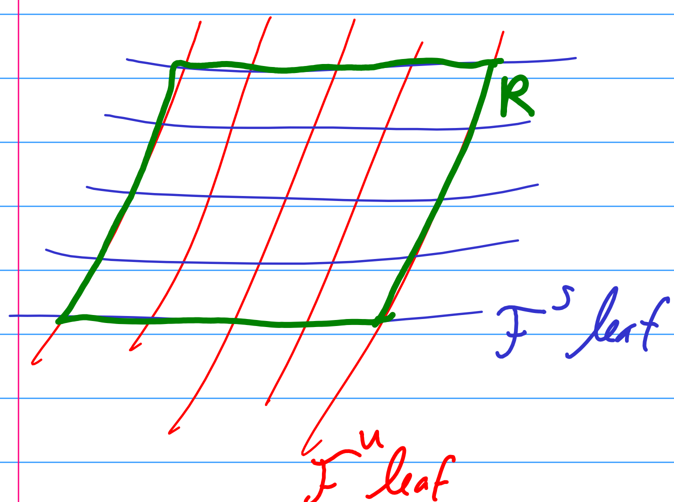
[It follows that $h(\sigma_A) \geq h(f)$]

and $\log(\operatorname{spr} A) = h(\sigma_A) = \log \lambda$.

This will close the sequence of inequalities and we conclude

$$\log \lambda = G_f = \gamma_{f\#} = h(f) = h(\sigma_A).$$

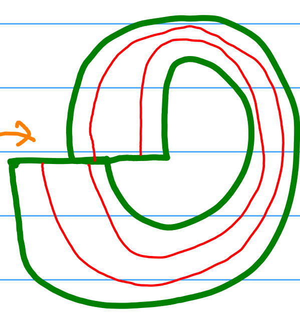
We will sketch the proof without being too precise about definitions. In particular, we avoid a precise def'n of a $(\mathcal{F}^s, \mathcal{F}^u)$-rectangle R: (or birectangle)



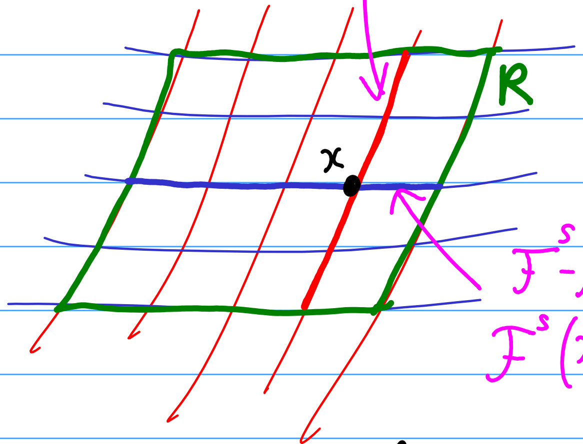
A birectangle is bounded by leaves of $\mathcal{F}^s$ and $\mathcal{F}^u$.
It is a map
$\varphi: I \times I \to M$ such that φ is an embedding for a "good" birectangle.

$[0,1]$

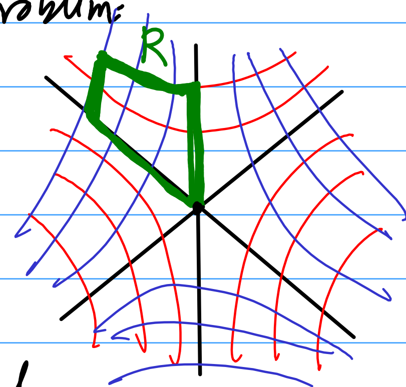
Bad:



Because the ends of the rectangle cross.



Singularities are not a problem:



For a good birectangle, $x \subset R$ is contained in only one fiber of $\mathcal{F}^s$ and one of $\mathcal{F}^u$.

The width of a birectangle is

$$\mathcal{W}(R) = \max\left(\mu^u(\mathcal{F}^s\text{-fiber}), \mu^s(\mathcal{F}^u\text{-fiber})\right).$$

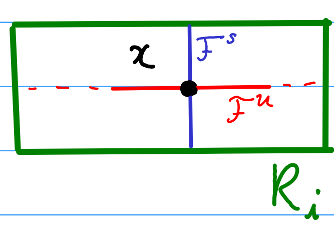
A Markov partition for the pA $f: M \to M$ is a collection of birectangles $\mathcal{R} = \{R_1, \dots, R_k\}$ s.t.

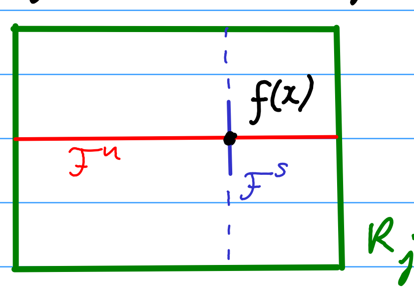
1) $\bigcup_{i=1}^{k} R_i = M$

2) R_i is a good birectangle

3) $\mathrm{Int}\, R_i \cap \mathrm{Int}\, R_j = \emptyset$, $i \neq j$

4) $\begin{matrix} x \in \mathrm{Int}\, R_i \\ f(x) \in \mathrm{Int}\, R_j \end{matrix} \Longrightarrow \begin{matrix} f(\mathcal{F}^s(x, R_i)) \subset \mathcal{F}^s(f(x), R_j) \\ f^{-1}(\mathcal{F}^u(f(x), R_j)) \subset \mathcal{F}^u(x, R_i) \end{matrix}$

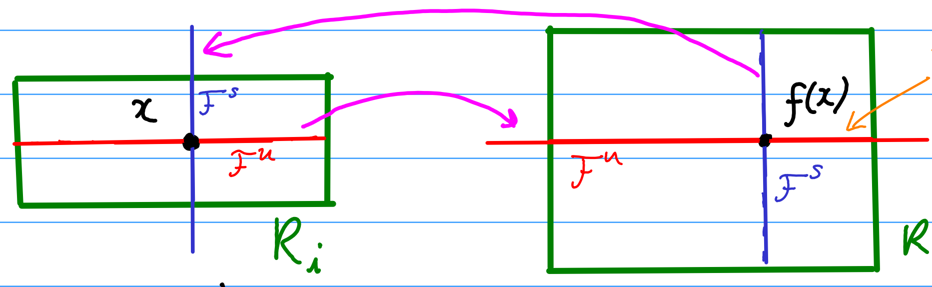




5) $x \in \mathrm{Int}(R_i)$, $f(x) \in \mathrm{Int}(R_j)$

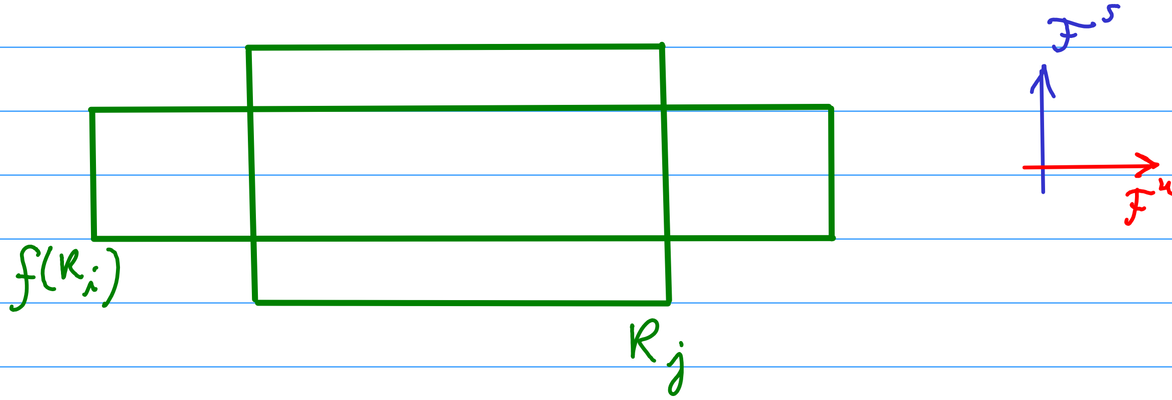
$$\Rightarrow f(\mathcal{F}^u(x, R_i)) \cap R_j = \mathcal{F}^u(f(x), R_j)$$

and $$f^{-1}(\mathcal{F}^s(f(x), R_j)) \cap R_i = \mathcal{F}^s(x, R_i)$$



The intersection of the image of the fibre $\mathcal{F}^u(x, R_i)$ with R_j is $\mathcal{F}^u(f(x), R_j)$.

This says that $f(R_i)$ goes across R_j only once:



We will discuss later how to construct a Markov partition for a pA.

Now here come the subshift of finite type Σ_A:

Let A be the $k \times k$ matrix defined by

$$a_{ij} = \begin{cases} 1, & \text{if } f(\mathrm{Int}\, R_i) \cap R_j \neq \emptyset \\ 0, & \text{otherwise.} \end{cases}$$

Just look at overlaps after one iteration!

Tricky part: if $\underline{b} \in \Sigma_A$, then

$$\bigcap_{i \in \mathbb{Z}} f^{-i}(R_{b_i}) \text{ contains exactly one point.}$$

To show this, we need:

Lemma: i) For $a_{ij} = 1$, $f(R_i) \cap R_j$ is a nonempty good birectangle which is a union of $\mathcal{F}^u$-fibers of R_j.

ii) If C is a birectangle contained in R_i which is a union of $\mathcal{F}^u$-fibers of R_i, then $f(C) \cap R_j$ is a non-empty birectangle which is a union of $\mathcal{F}^u$-fibers of R_j

iii) Given $\underline{b} \in \Sigma_A$, $\forall n \in \mathbb{N}$, $\bigcap_{i=-n}^{n} f^{-i}(R_{b_i})$ is a nonempty birectangle, with

$$\mathcal{W}\left(\bigcap_{i=-n}^{n} f^{-i}(R_{b_i})\right) \leq \lambda^{-n} \max\{\mathcal{W}(R_1), \dots, \mathcal{W}(R_k)\}$$

Proof: i) and ii) are straightforward and follow from the definition of birectangle. [FLP p. 198]

Using ii) repeatedly, it follows that each set of the form

$$f^n(R_{b_i}) \cap f^{n-1}(R_{b_{i+1}}) \cap \cdots \cap f(R_{b_{i+n-1}}) \cap R_{b_{i+n}}$$

is a non-empty birectangle which is a union of $\mathcal{F}^u$-fibers of $R_{b_{i+n}}$, since $\mathrm{Int}(f(R_{b_i})) \cap \mathrm{Int}(R_{b_{i+1}}) \neq \emptyset$.

$$\cdots \cap f(f^{n-2}(R_{b_{-n+1}})) \cap f^{n-2}(R_{b_{-n+2}}) \cap \cdots$$

In particular,

$$f^{n}(R_{b_{-n}}) \cap f^{n-1}(R_{b_{-n+1}}) \cap f^{n-2}(R_{b_{-n+2}}) \cap \cdots$$
$$\cdots \cap f^{-n+1}(R_{b_{n-1}}) \cap f^{-n}(R_{b_n})$$
$$= \bigcap_{i=-n}^{n} f^{-i}(R_{b_i})$$

is a nonempty birectangle in $f^{-n}(R_{b_n}) \subset R_{b_0}$.

If we consider an $\mathcal{F}^u$-fiber in R_{b_0}, it is getting shorter by a factor λ^{-1} under f^{-1}. $\mathcal{F}^s$-fibers get shorter under f. Hence,

$$\mathcal{W}\left(\bigcap_{i=-n}^{n} f^{-i}(R_{b_i})\right) \leq \lambda^{-n} \max\{\mathcal{W}(R_1), \dots \mathcal{W}(R_k)\}.$$

which completes the proof of iii).

By the lemma, the set $\bigcap_{i\in\mathbb{Z}} f^{-i}(R_{b_i})$ is the intersection of a decreasing sequence of nonempty sets.

Hence, $\bigcap_{i\in\mathbb{Z}} f^{-i}(R_{b_i})$ is non void. It is reduced to one point because $\mathcal{W}\left(\bigcap_{i=-n}^{n} f^{-i}(R_{b_i})\right) \to 0$ as $n \to \infty$.

Now we can define the map $\theta : \Sigma_A \to M$ by

$$\theta(\underline{b}) = \bigcap_{i \in \mathbb{Z}} f^{-i}(R_{b_i}).$$

$$\begin{tikzcd} \Sigma_A \arrow[r, "\sigma_A"] \arrow[d, "\theta"'] & \Sigma_A \arrow[d, "\theta"] \\ M \arrow[r, "f"] & M \end{tikzcd}$$

It is continuous, and $\theta \sigma_A = f \theta$.

We will postpone showing surjectivity of θ to the end.

We need to show that:

spectral radius

$$\operatorname{spr} A = \lambda$$ ← the dilatation of the pA.

Put $y_i = \mu^u(\mathcal{F}^s\text{-fiber of } R_i)$. Since $\mathcal{F}^s$ fibers are bounded by $\mathcal{F}^u$-fibers, y_i is independent of the choice of fiber in R_i. Also, $y_i > 0$.

If we follow our fiber under one iteration of f^{-1}:

$$y_j = \sum_{i=1}^{k} \frac{y_i}{\lambda} a_{ij}$$

This is basically "conservation of $\mathcal{F}^s$-width": we sum over birectangles R_i such that $f(R_i) \subset R_j$.

Each rectangle narrows by λ, and they don't overlap. Hence,

$$\lambda y_j = \sum_{i=1}^{k} y_i a_{ij}$$

λ is an eigenvalue of A.

$$\geq \left(\sum_{i=1}^{k} a_{ij} \right) \min_i y_i$$

$$\lambda \sum_j y_j \geq \|A\| \min_i y_i$$

$$\lambda^n \sum_j y_j \geq \|A^n\| \min_i y_j$$

$$\lambda \geq \|A^n\|^{1/n} \left(\frac{\min(y_1, \cdots, y_k)}{\sum_j y_j} \right)^{1/n}$$

Since $y_i \geq 0$, $\min_i y_i \geq 0$, and $\lim_{n\to\infty} \left(\frac{\min(y_1, \cdots, y_k)}{\sum_j y_j} \right)^{1/n} = 1.$

Hence,

$$\lambda \geq \lim_{n\to\infty} \|A^n\|^{1/n} = \operatorname{spr} A.$$

Since λ is an eigenvalue of A, $\lambda = \operatorname{spr} A$.

Now for surjectivity of θ: observe that the closure of $\operatorname{Int}(R_i) = R_i$, $1 \leq i \leq k$. Hence,

$$V = \bigcup_{i=1}^{k} \operatorname{Int} R_i \quad \text{is a dense open set in } M.$$

By the Baire category theorem,

$$U = \bigcap_{i \in \mathbb{Z}} f^{-i}(V) \quad \text{is dense in } M.$$

For $x \in U$, $\forall n \in \mathbb{Z}$, the point $f^n(x)$ is in a unique $\operatorname{Int}(R_{b_n})$, and $\underline{b} = \{b_n\}_{n\in\mathbb{Z}}$ is an element of Σ_A.

Clearly, $\theta(\underline{b}) = x$. Thus $\theta(\Sigma_A) \supset U$.
As Σ_A is compact and θ continuous, we have $\theta(\Sigma_A) = M$.