Braids Lecture 30: Markov Partition for pA's

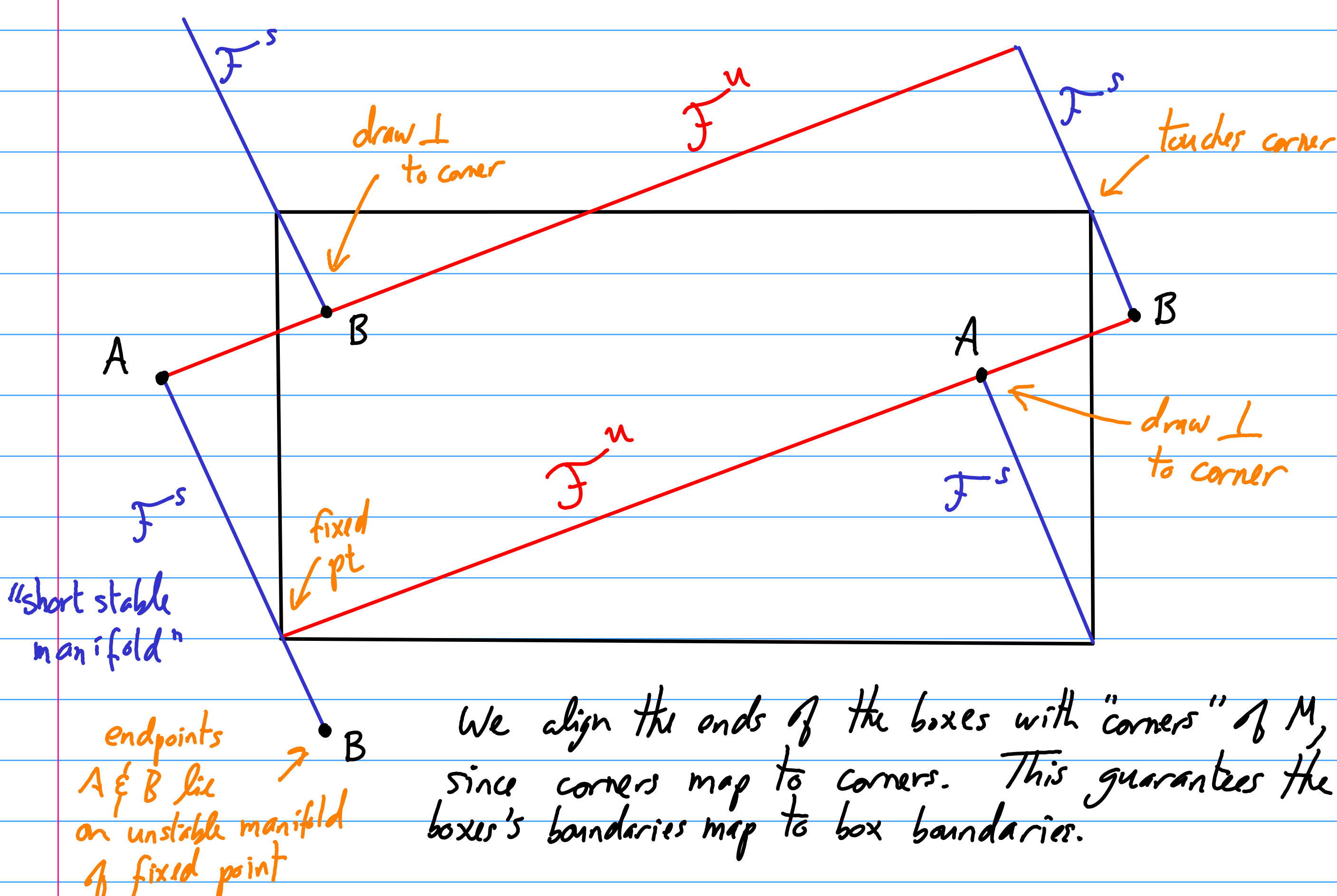
In the proof that $h(f) = \log \lambda$, for $f: M \to M$ pseudo-Anosov, we assumed (crucially!) that f had a Markov partition. We now sketch the demonstration that it's always possible to do this.

As always, we start with the Anosov diffeo $f: T^2 \to T^2$:

$$f(\underline{x}) = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} \bmod 1, \qquad \underline{x} = \begin{pmatrix} x \\ y \end{pmatrix}$$

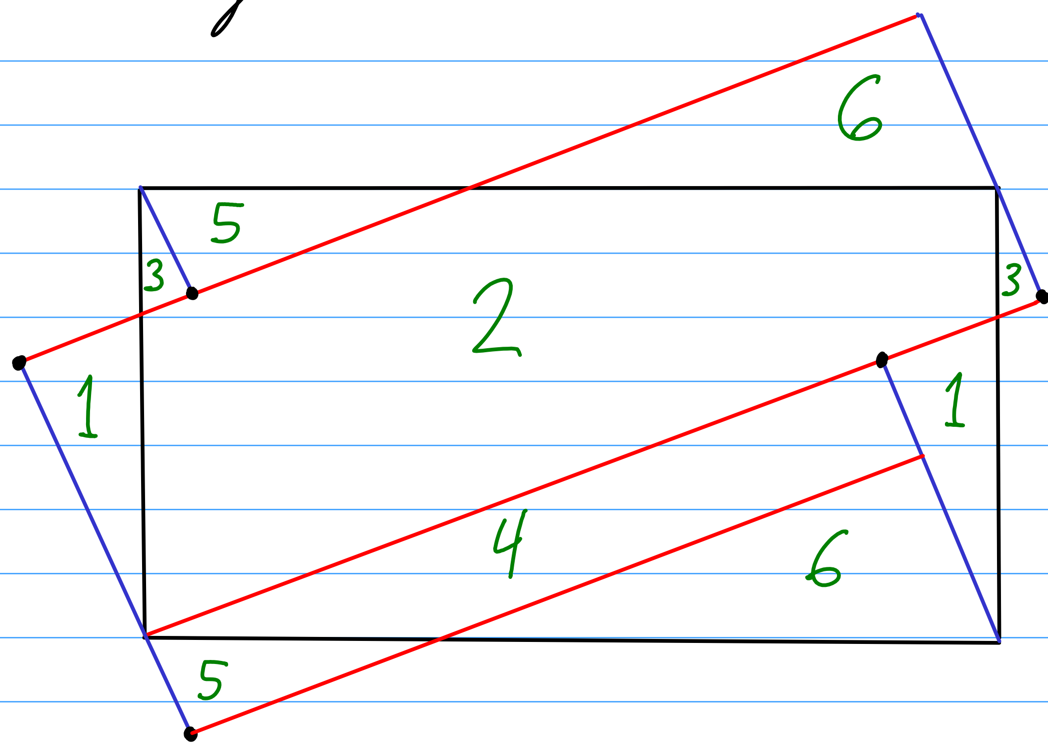
Here $\lambda = \frac{1}{2}(3+\sqrt{5})$, $\lambda^{-1} = \frac{1}{2}(3-\sqrt{5})$.

Fundamental domain: draw eigenspaces from the origin

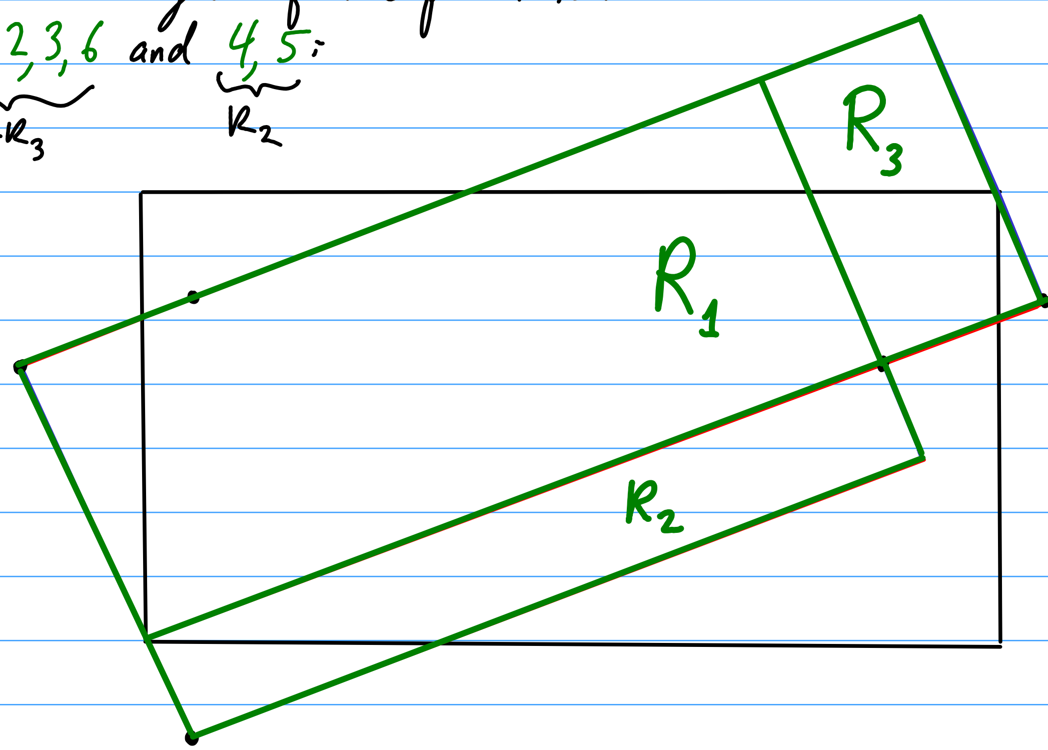


We align the ends of the boxes with "corners" of M, since corners map to corners. This guarantees the boxes's boundaries map to box boundaries.

Label the triangles:



The three rectangles of the partition are $\underbrace{1,2,3,6}_{R_1+R_3}$ and $\underbrace{4,5}_{R_2}$:



Under f, these boxes are stretched and squashed, but they maintain their alignment with $\mathcal{F}^u$ and $\mathcal{F}^s$ fibres.

[See computer figures at the end of this lecture.]

The topological transition matrix is

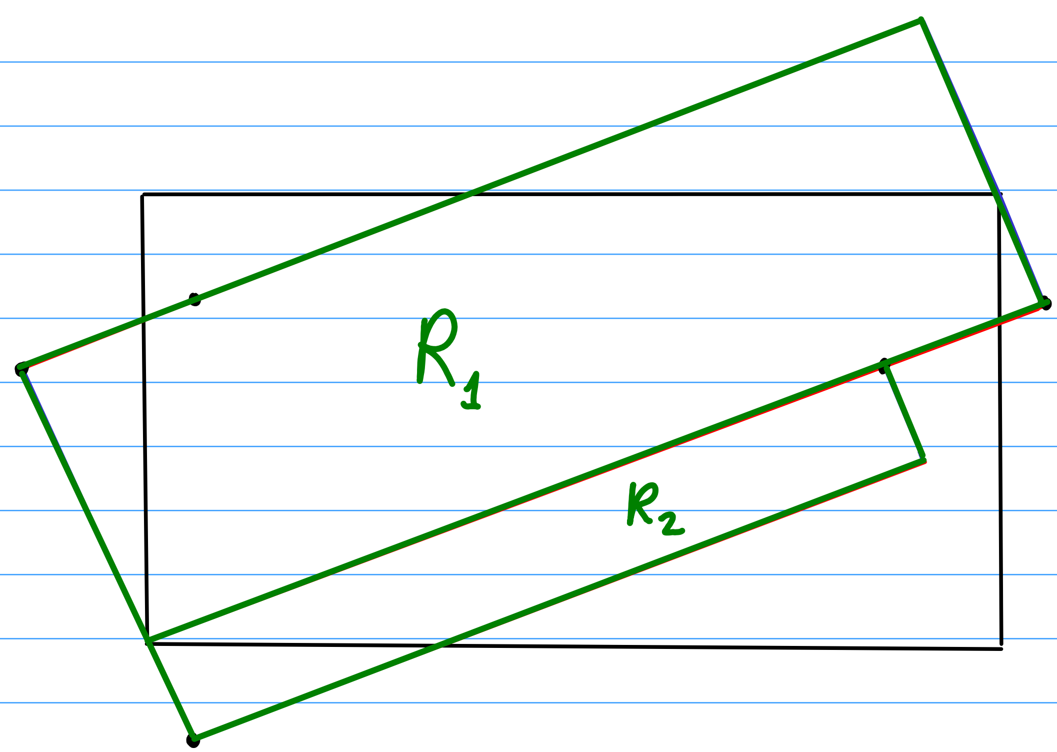
$$A = \begin{array}{c} \\ \\ \\ \end{array}\begin{array}{ccc} R_1 & R_2 & R_3 \end{array}$$

$$A = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 0 & 1 \end{pmatrix} \begin{array}{l} f(R_1) \\ f(R_2) \\ f(R_3) \end{array}$$

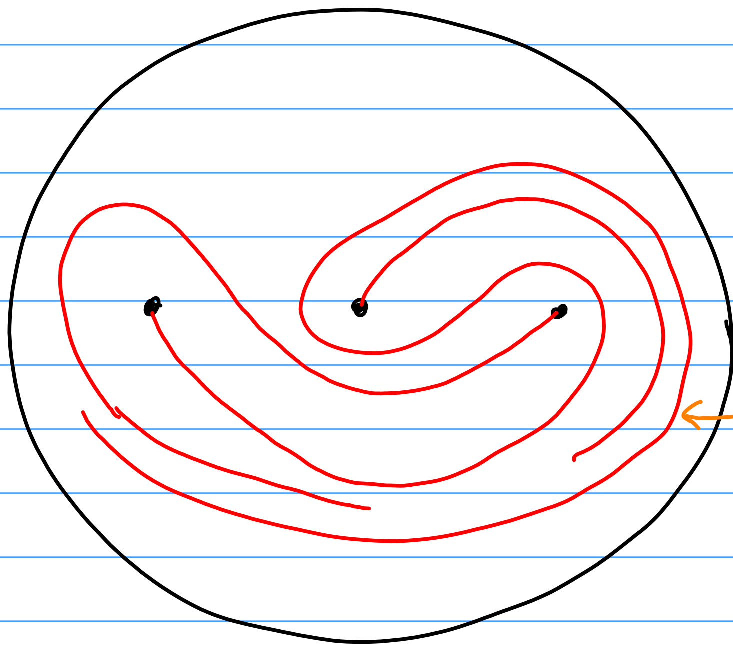
As expected, $\operatorname{spr} A = \frac{1}{2}(3+\sqrt{5})$.

Note also that we can drop box 3, since $R_1' = R_1 \cup R_3$ is still a good birectangle, if we relax the requirement that rectangles intersect only once. If we use only two boxes, we get $A' = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}$, the same matrix as the pA.

Another way to more systematically generate a Markov partition is to start with R_1, a rectangle obtained from completing F^u to the "short stable manifold", and then taking intersections with direct and inverse images.



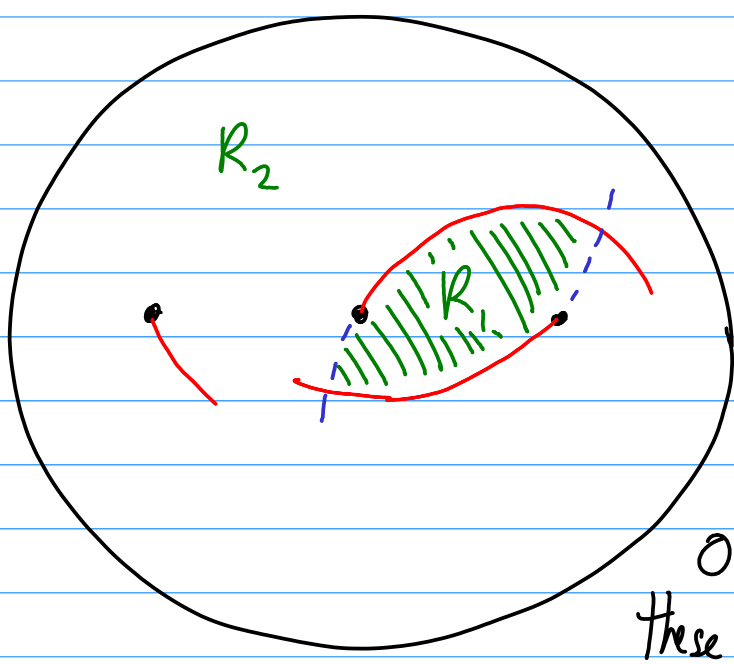
To show how to build the Markov partition of a pseudo-Anosov consider the structure of the foliation for the diffeomorphism on the disc with 3 punctures associated with the braid $\sigma_1 \sigma_2^{-1}$:



[See picture at the end of the notes.]

unstable manifolds of punctures

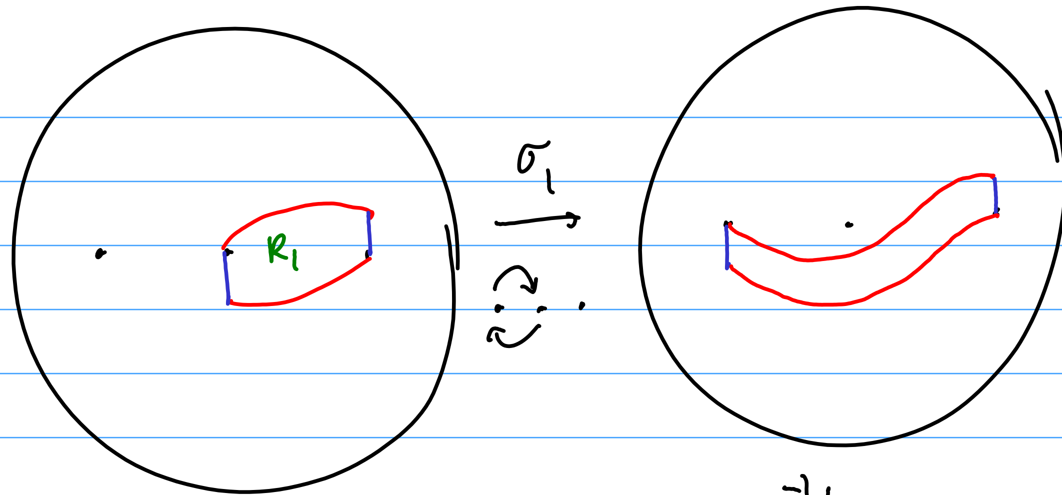
Near the 1-prongs, we have a structure that resembles:



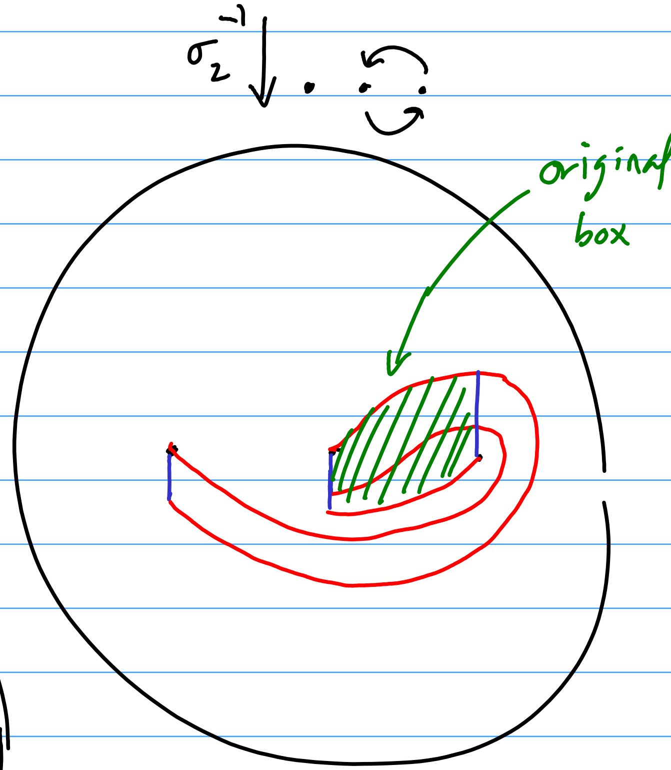
R_1 and R_2 are both good birectangles. (Don't be fooled by the apparent tangency at the singularities: not a problem)

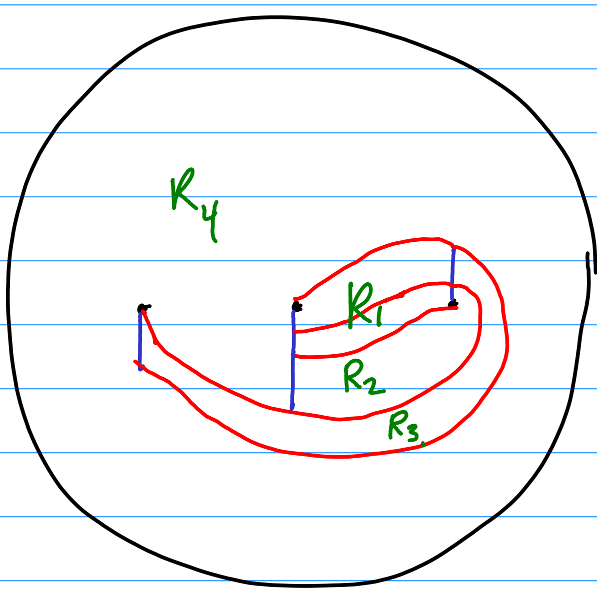
$$M = R_1 \cup R_2.$$

Of course, we don't know whether these form a Markov partition.



Clearly, the $\mathcal{F}^u$-fibers of R_1 do not fit "snugly" in the Markov boxes. Hence, we refine the boxes by taking intersections.





Intersecting with enough forward/backward iterates, we eventually obtain a Markov partition. Note that in general it will have way more boxes than are necessary.