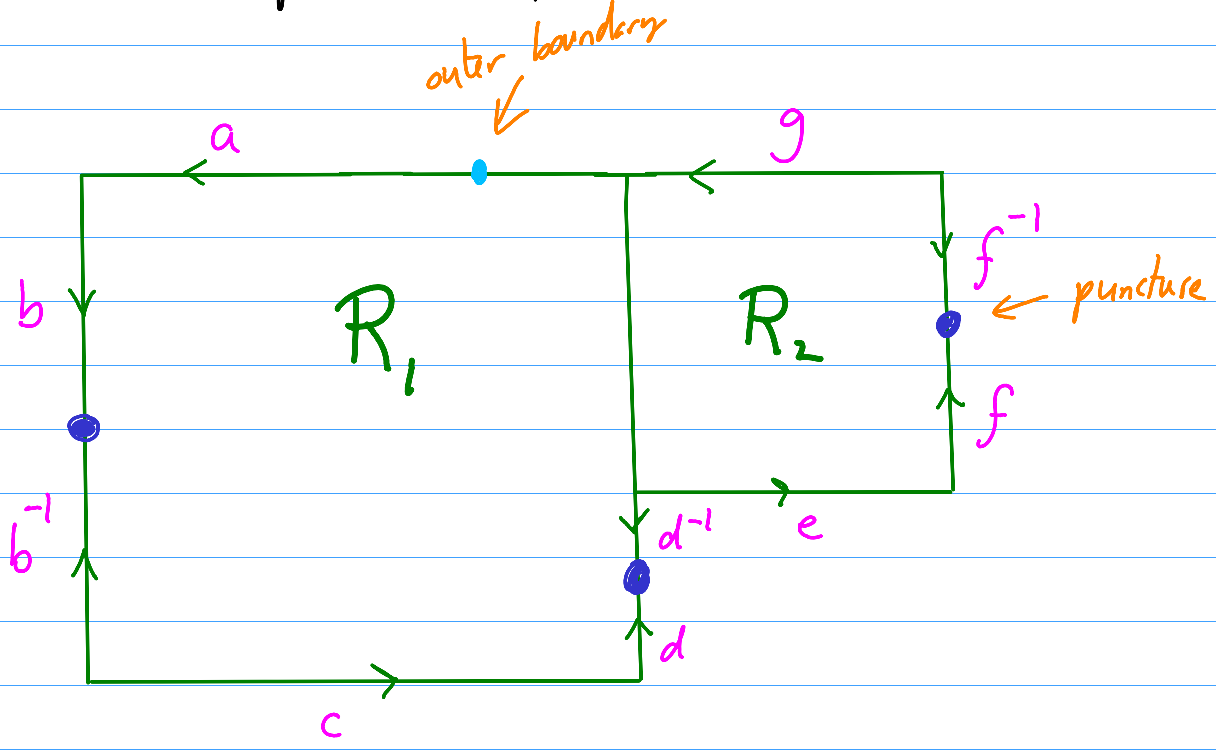
Braids Lecture 31: From Markov boxes to train tracks

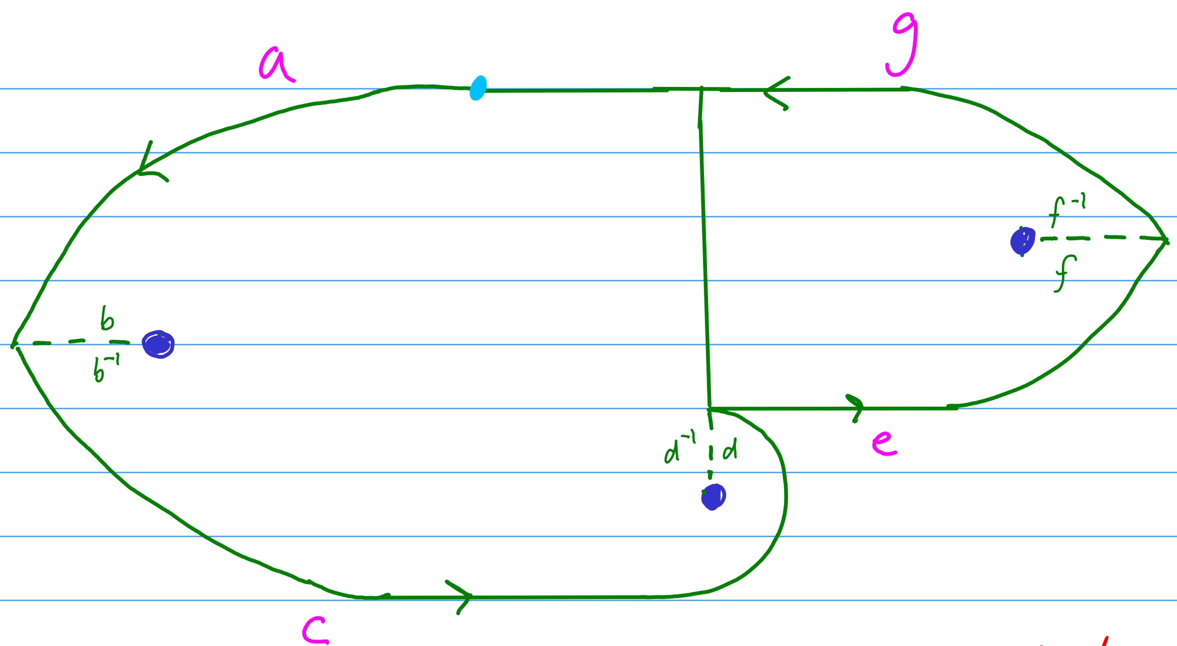
Consider the Markov partion associated with a pA on the disc with three punctures. This is the same as a sphere with 4 punctures, with one of the punctures (the disc's outer boundary) held fixed.

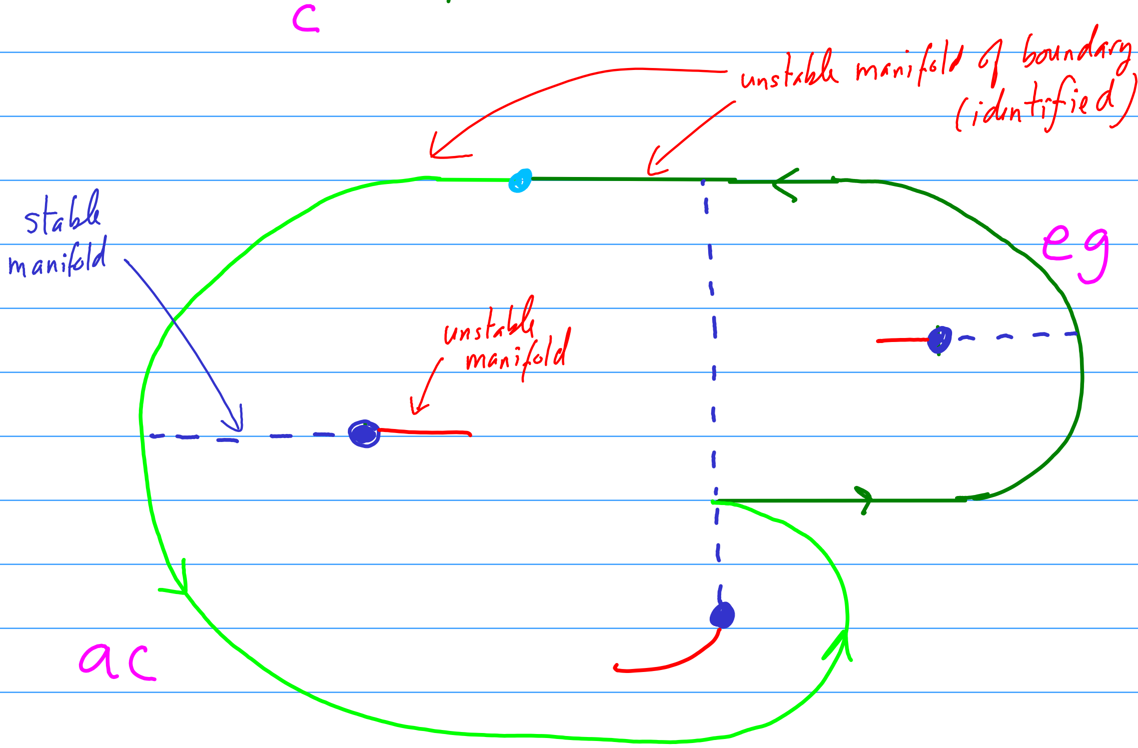
We describe the partition as follows:



with the identification $ac = g^{-1}e^{-1}$.

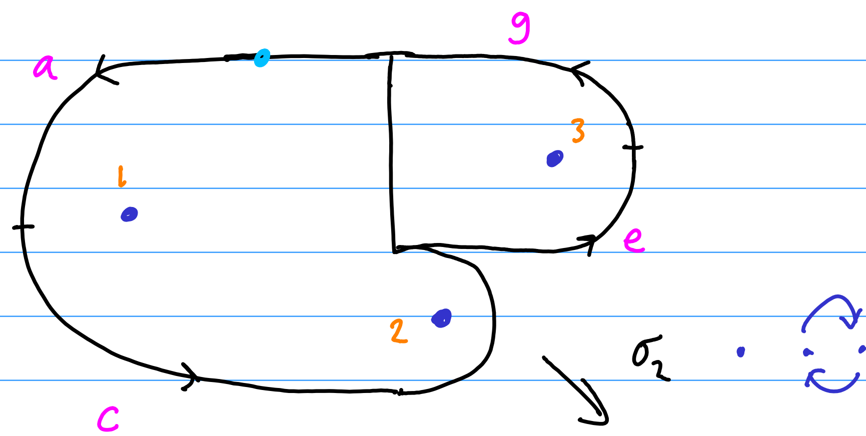
To see that this is a sphere, first glue the identified edges:

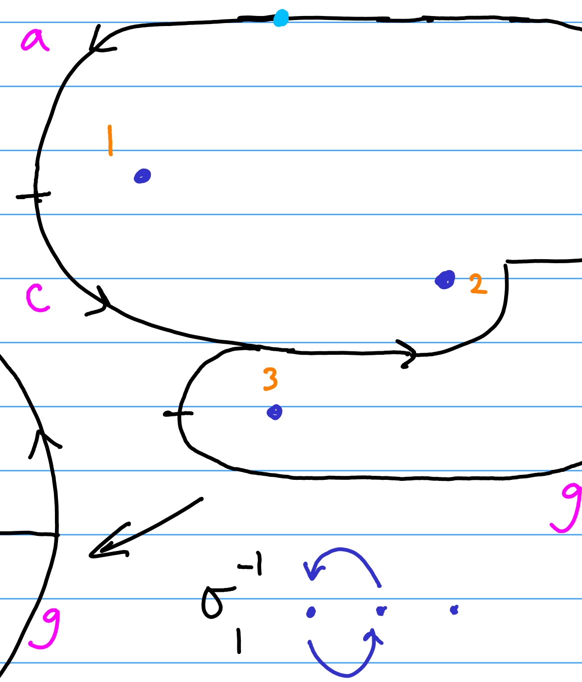


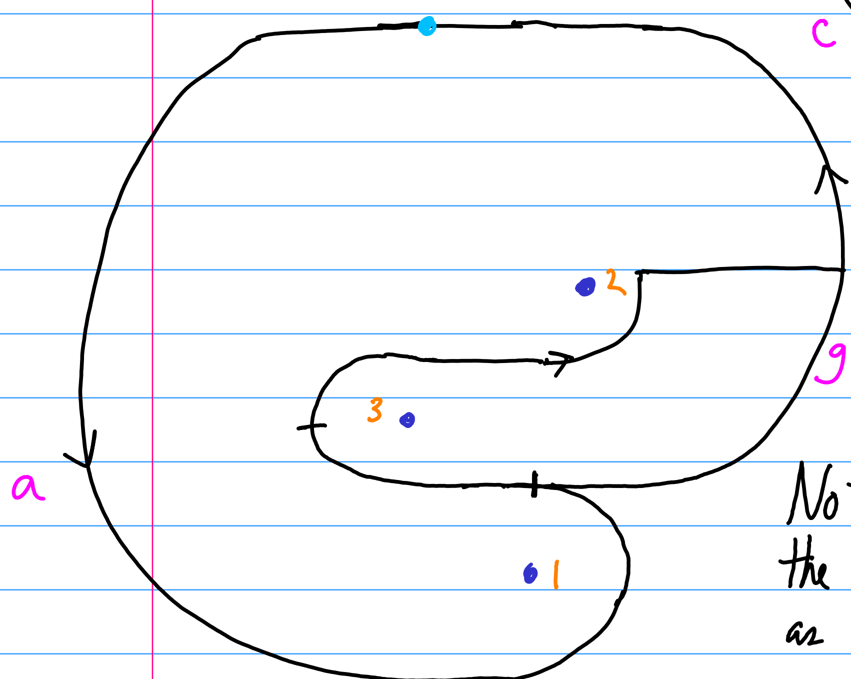


Gluing ac on $(eg)^{-1}$ then closes the "pouch", giving us sphere.

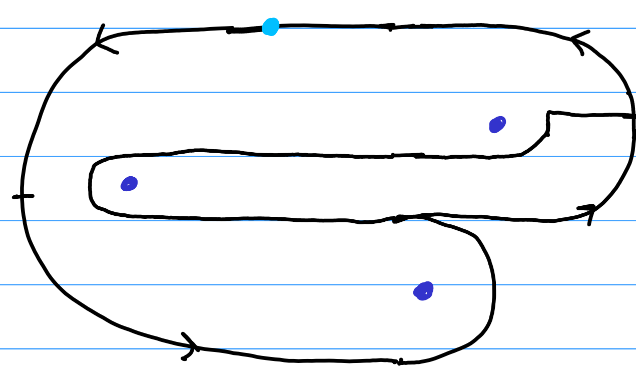
Now, the diffeomorphism we have in mind is the pA associated with the braid $\sigma_2 \sigma_1^{-1}$.







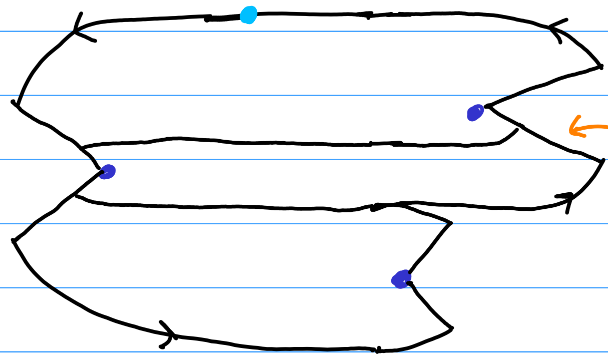
Notice how this has the same general "shape" as before! In fact, it fits snugly:



a and g^{-1} are identified.

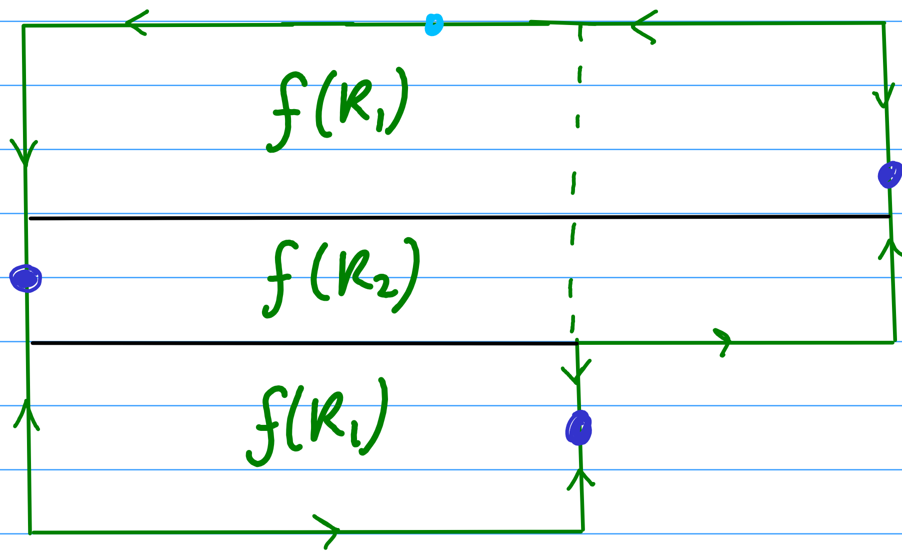
Notice that stable manifolds have mapped to stable manifolds!

"pacmanize"



Now cut again to make the rectangles explicit.

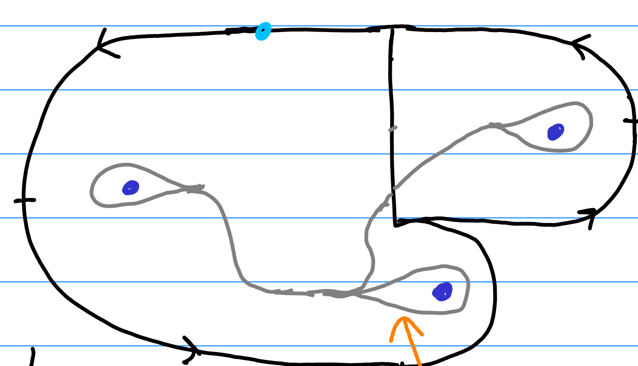
Notice that the boundary between the boxes mapped to a boundary → good Markov partition.



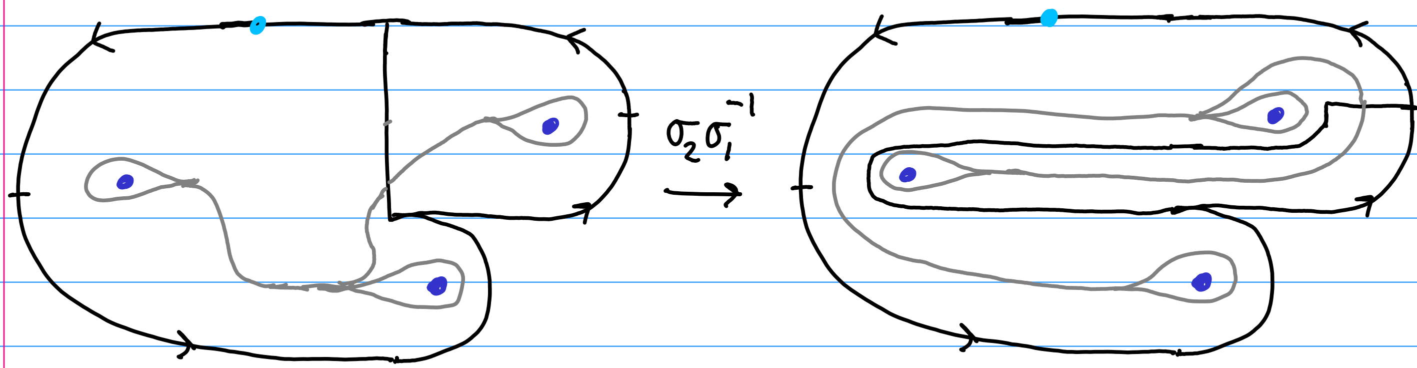
If everything is done correctly, each box has expanded its length by λ, shrunk its width by λ^{-1}!

Looking at the overlaps with the original rectangles, we see that the Markov transition matrix is $\begin{pmatrix}2 & 1\\ 1 & 1\end{pmatrix}$, as expected.

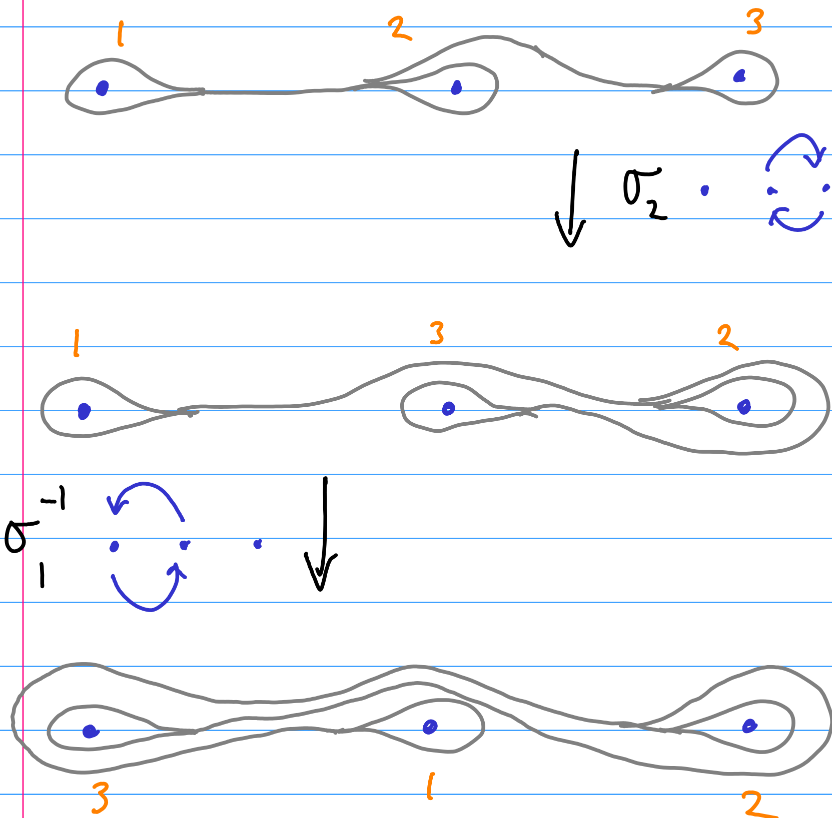
Now, the crucial observation is that the changes in width and length of boxes are not independent. Consider the one-dimensional branched manifold overlayed on the rectangles, called a train track representative (or simply train track) for the Markov partition.



train track!

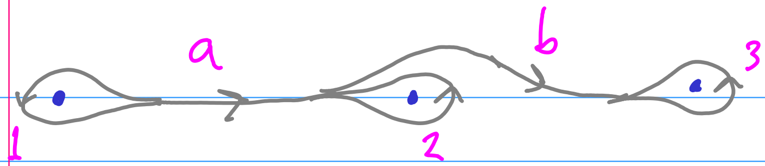


The train track must satisfy a similar invariance property to the Markov boxes:



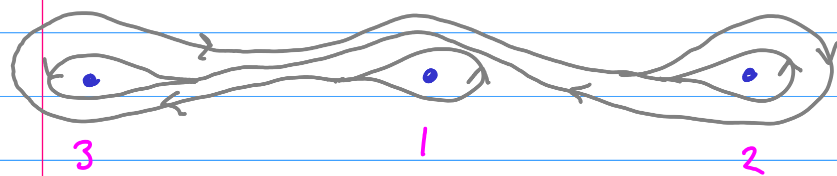
Notice how the final picture "looks" like the initial branched manifold.

We can use this to calculate λ. To do this, we label and orient the edges of the train track.



Label "main" edges by letters, and "loops" by numbers.

$\downarrow \; \sigma_2 \sigma_1^{-1}$



Now, much like Markov boxes, described the transformed edges in terms of edge paths, that is, a sequence of original edges.

Get the train track map:

$$
\begin{aligned}
a &\mapsto \bar{a}\,\bar{1}\,a\,b \\
b &\mapsto \bar{3}\,\bar{b}\,\bar{a} \\
1 &\mapsto 2 \\
2 &\mapsto 3 \\
3 &\mapsto 1
\end{aligned}
$$

$\bar{a}$ means a^{-1}, with respect to direction

The loops are simply permuted

If we find we cannot write the images of the edges in terms of original edge paths, we have the wrong train track.

Example: 

Cannot be written in terms of 

We say that the train track supports the pA.

How to get λ: we Abelianize: only count the occurrences of each edge in the map:

$$\begin{pmatrix} a \\ b \\ 1 \\ 2 \\ 3 \end{pmatrix} \longmapsto \left(\begin{array}{cc|ccc} 2 & 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 & 1 \\ \hline 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 \end{array}\right) \begin{pmatrix} a \\ b \\ 1 \\ 2 \\ 3 \end{pmatrix}$$

permutation matrix
$\to$ |eigenvalues| = 1

The spectral radius of the matrix is the spectral radius of $\begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}$.

Since loops map only to loops, the transition matrix always has this block triangular form.

More generally, things are of course not so simple. Here are the steps:

1) Find an invariant train track for the diffeo.

2) Verify that the TT map is efficient (no cancellations $\to$ see later)

3) Verify that the transition matrix (the sub-block of main edges) is irreducible.

We will look at more complicated examples to illustrate these steps.