Braids Lecture 32: Train Track Graphs

A graph Γ is a collection of edges and vertices. Each edge either connects two vertices, or a vertex to itself.

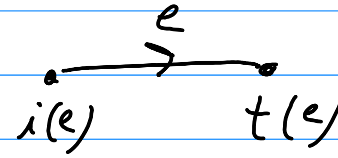
$E(\Gamma)$ is the set of edges of Γ.

$V(\Gamma)$ is the set of vertices of Γ.

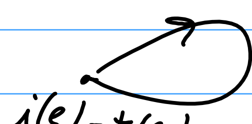
By convention, edges are closed sets. Γ is the union of $E(\Gamma)$ and $V(\Gamma)$. [Some authors use open edges]

Each edge is given an arbitrary orientation. The edge e but with opposite orientation is denoted $\bar{e}$. [reverse of e]
[Note: $\bar{e} \in E(\Gamma)$]

$i(e)$ = initial vertex of e
$t(e)$ = final (terminal) vertex of e



For any vertex $v \in \Gamma$, the star of v is the set



Differs from Stallings

$$St(v,\Gamma) = \{ e \in E(\Gamma) \mid i(e) = v \text{ or } t(e) = v \}.$$

$\mathrm{card}\, St(v,\Gamma) = \mathrm{val}(v)$ is the valence or degree of vertex v in Γ.

An edge path $\gamma = e_1 e_2 \cdots e_m$ is an ordered list of edges such that

$$t(e_k) = i(e_{k+1}), \quad 1 \leq k \leq m-1.$$

An edge path is <u>reduced</u> if $e_k \neq \bar{e}_{k+1}$, $1 \leq k \leq m-1$.

It is <u>closed</u> if $t(e_m) = i(e_1)$. The trivial path based at v is denoted 1_v.

A graph is <u>connected</u> if there is an edge path between any two vertices. A connected graph with no nontrivial reduced paths is called a <u>tree</u>.

At each vertex $v \in \Gamma$, $St(v, \Gamma)$ can be given a <u>cyclic ordering</u>. Two edges in $St(v, \Gamma)$ are <u>adjacent</u> if they are consecutive in the cyclic ordering.

For two graphs Γ and Γ', $f : \Gamma \to \Gamma'$ is a <u>graph map</u> if it is continuous and

1. A vertex in Γ is mapped to a vertex in Γ';
2. An edge in Γ is mapped to an edge path in Γ';
3. $f|_{\mathrm{Int}\, e}$ is locally injective, $\forall e \in \Gamma$.

The map f induces a map $f_v : St(v, \Gamma) \to St(f(v), \Gamma')$.

If f_v is injective $\forall v \in \Gamma$, f is an <u>immersion</u>.

[See the classic paper by Stallings (1983)]

If $E(\Gamma) = \{e_i\}$ and $E(\Gamma') = \{e_i'\}$, then $f(e_i)$ is an edge path in Γ'.

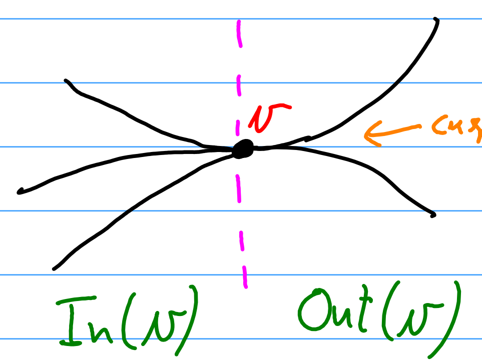
The incidence matrix M^f of f contains nonnegative integers such that

$$M^f_{ij} = \#\text{ of occurences of } e_j' \text{ and } \bar{e}_j' \text{ in edge path } f(e_i).$$

Train Tracks: Let τ be a graph embedded in a surface S. We say that τ has a smooth structure if at each $v \in \tau$, there is a partition of $St(v, \tau)$ in two nonempty subsets:

$In(v)$ = incoming edges, $Out(v)$ = outgoing edges

In addition, all the edges have a common tangent at v.

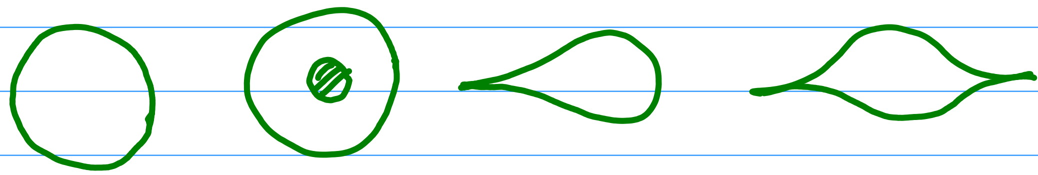


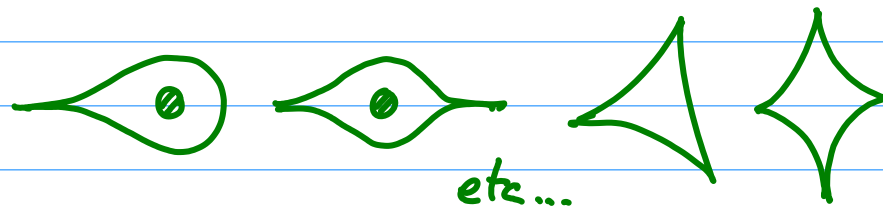
The sets $In(v)$ and $Out(v)$ are interchangeable.

A cusp is the "corner" formed by a pair $\{e_i, e_j\}$ of adjacent edges at v, with both edges in $In(v)$ or $Out(v)$.

Let $h : \tau \to S$ be an embedding. The pair (τ, h) is a <u>train track</u> if

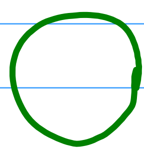
1. τ has no vertices of valence 1 or 2.
2. h preserves the smooth structure of τ
3. The connected component of $S - h(\tau)$ are <u>not</u> of the type:

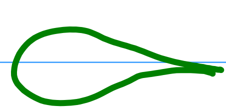


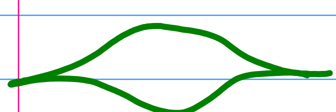
but rather are 

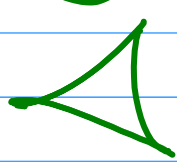
etc...

To make this last requirement more precise: take a connected component of $S - h(\tau)$. "Double" it by attaching a copy of it along its boundary, creating a "pouch". If we treat the corners of the pouch as boundary components, then the resulting surface must have negative Euler characteristic.

 **Nullgon**

 **Monogon**

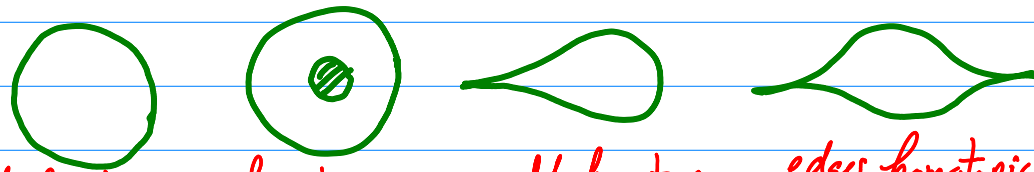
 **Bigon**

 **Trigon**

So $S - h(\tau)$ does not contain unpunctured nullgons, monogons, or bigons, or once-punctured nullgons.

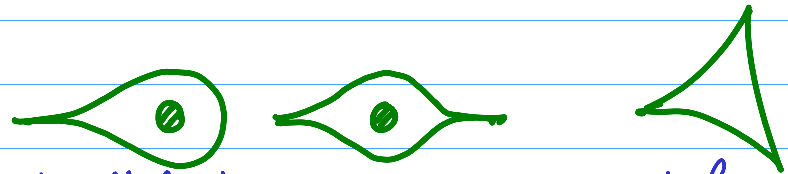
Why requirement 3? It implies that the edges of the train track are not null-homotopic, homotopic to a boundary, or to each other (holding the vertices fixed).

BAD



null-homotopic | homotopic to boundary | null-homotopic | edges homotopic to each other

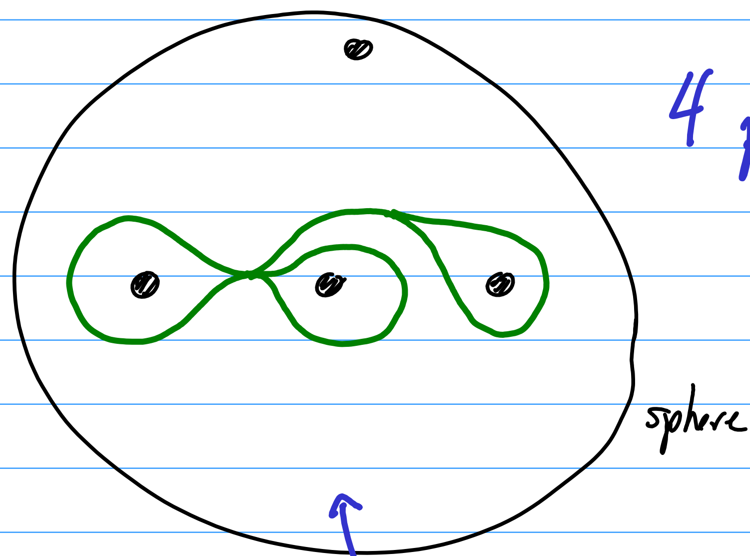
GOOD



Not null-homotopic because of boundary

Can't homotope w/o destroying smooth structure.

Example:



4 punctured monogons

4 th monogon! (Think about it...)