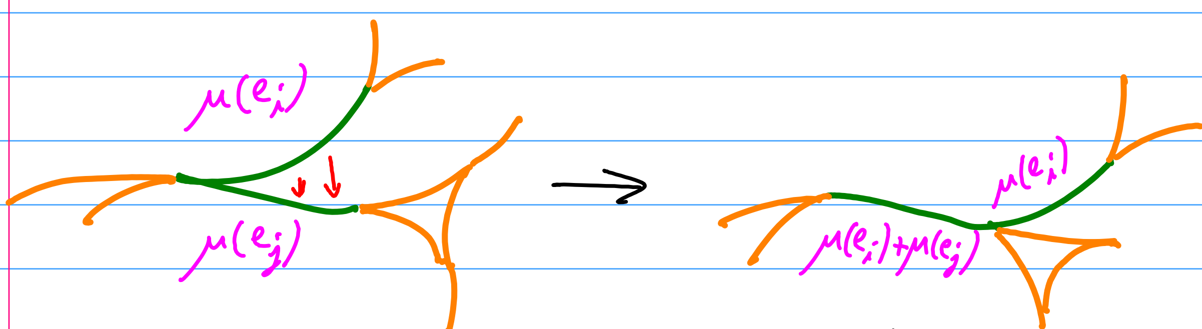
Braids Lecture 34: Measured Train Tracks and Fibered Neighbourhoods

Another structure we can put on the train track is a measure
$$\mu : E(\tau) \longrightarrow \mathbb{R}_+$$
satisfying
$$\mu(e_1^{In}) + \cdots + \mu(e_m^{In}) = \mu(e_1^{Out}) + \cdots + \mu(e_n^{Out})$$
where $e_i^{In} \in In(v)$ and $e_i^{Out} \in Out(v)$, at each vertex v. This is known as a switch condition.

The pair (τ, μ) [really (τ, ρ, μ)] is called a measured train track.

This is somewhat complementary to a length. We can also see how it transforms under an elementary folding:



This is less restrictive than a length: we can fold e_i on e_j or e_j on e_i, regardless of length. We can also fold past infinitesimal edges, as before.

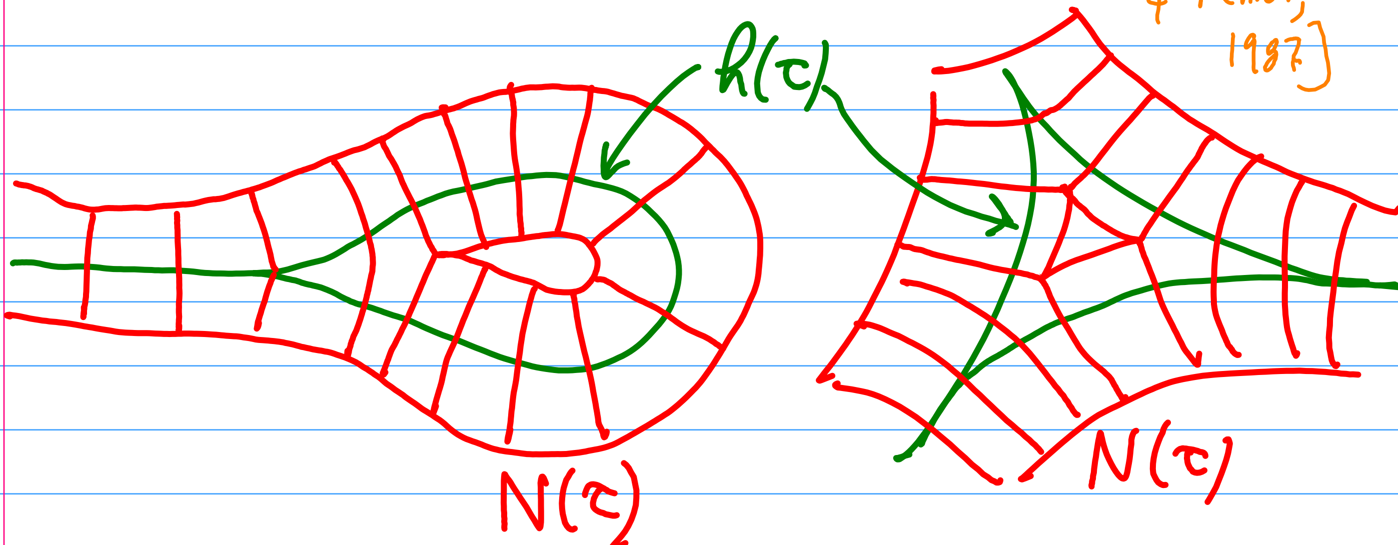
Note that the swith conditions are still satisfied after folding.

The class of a train track is the set of all train tracks that can be reached by elementary foldings (+ immersion).

Now let (τ, h) be a train track embedded in a surface S. Consider a regular neighbourhood $N(\tau) \subset S$ of $h(\tau)$ such that there is a retraction

$$N(\tau) \searrow h(\tau)$$

[Papadopoulos & Penner, 1987]

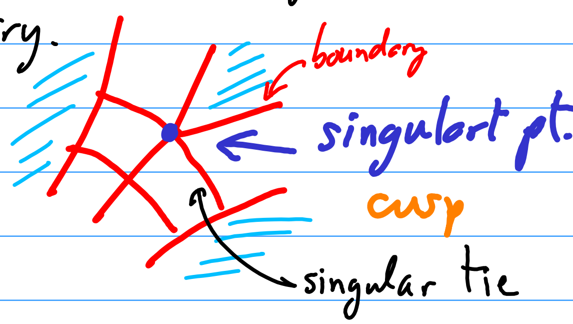


$N(\tau)$ is called a fibered neighbourhood of τ.

The fibers (and their transverse) of $N(\tau)$ form a foliation of $N(\tau)$.

A partial measured foliation of S is a foliation of a subsurface $\subset S$ that has all of its singular points on its boundary.

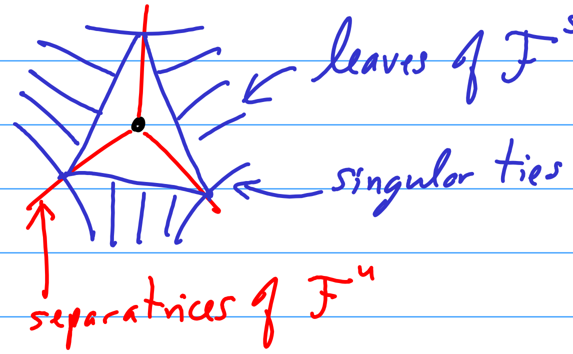
The set of cusps is in 1-1 correspondence with the set of vertices of τ.



A foliation $\mathcal{F}$ is carried by τ if $\mathcal{F}$ can be represented by a partial measured foliation contained in a fibered neighbourhood $N(\tau)$, respecting transversality.

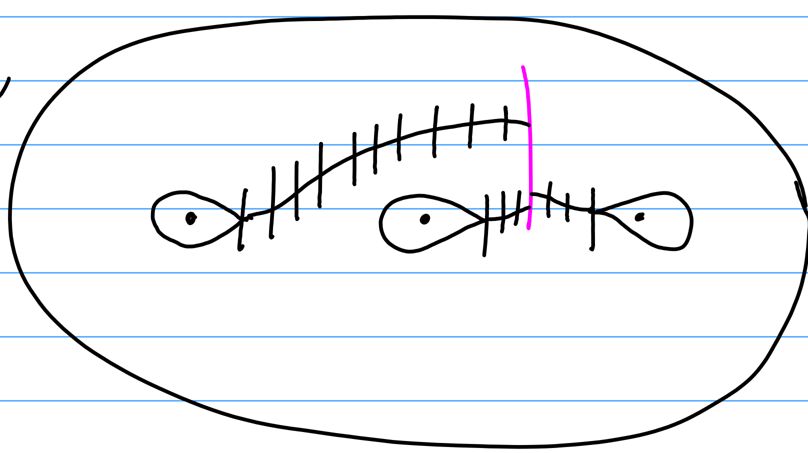
The construction of a fibered neighbourhood from foliations $(\mathcal{F}^u, \mathcal{F}^s)$ is straightforward [See Papadopoulos & Penner 1987]

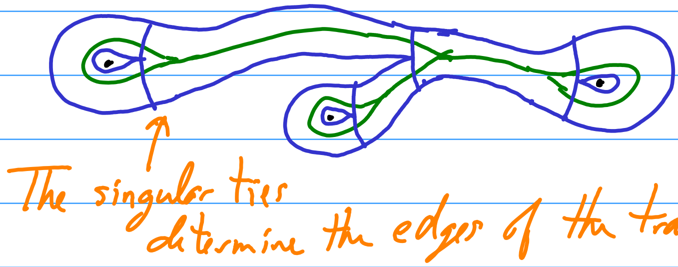
1. Remove a neighbourhood around each interior singularity (not on ∂_0):



2. Extend the separatrices of $\mathcal{F}^u$ at each singularity until they intercept an arbitrary curve transverse to $\mathcal{F}^u$

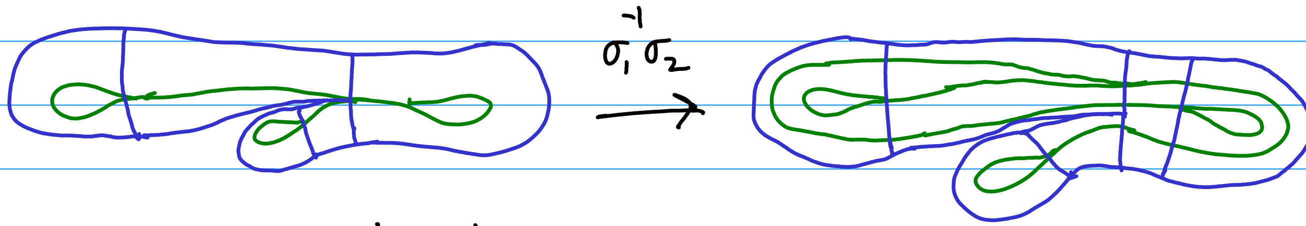
This must happen eventually, since each leaf is dense in S.





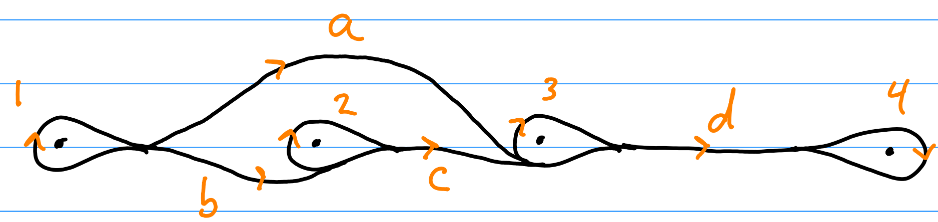
Can then retract along the fibers to get a train track!

The fibered neighbourhood also allows us to make sense of the image of a train track under a homeo:

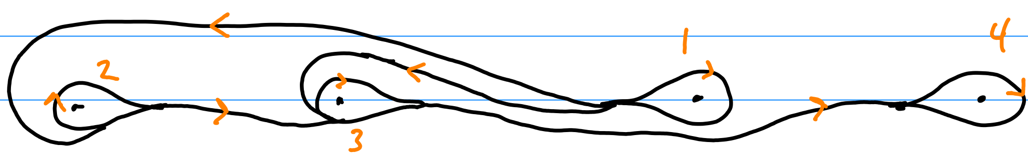


We say that this train track, and/or its fibered neighbourhood, <u>supports</u> the pA $\sigma_1^{-1}\sigma_2$.

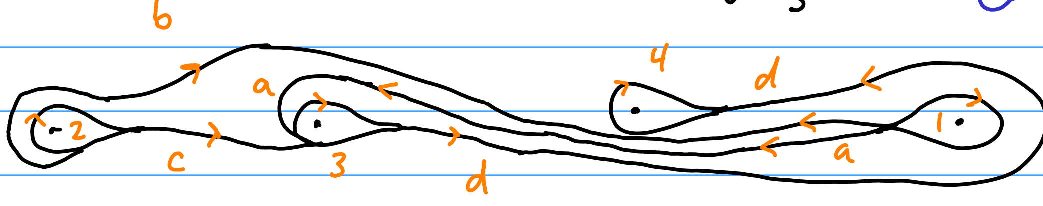
Again, we can build the incidence or transition matrix. Let's to a more complicated example, for the braid $\sigma_1\sigma_2\sigma_3^{-1}$.



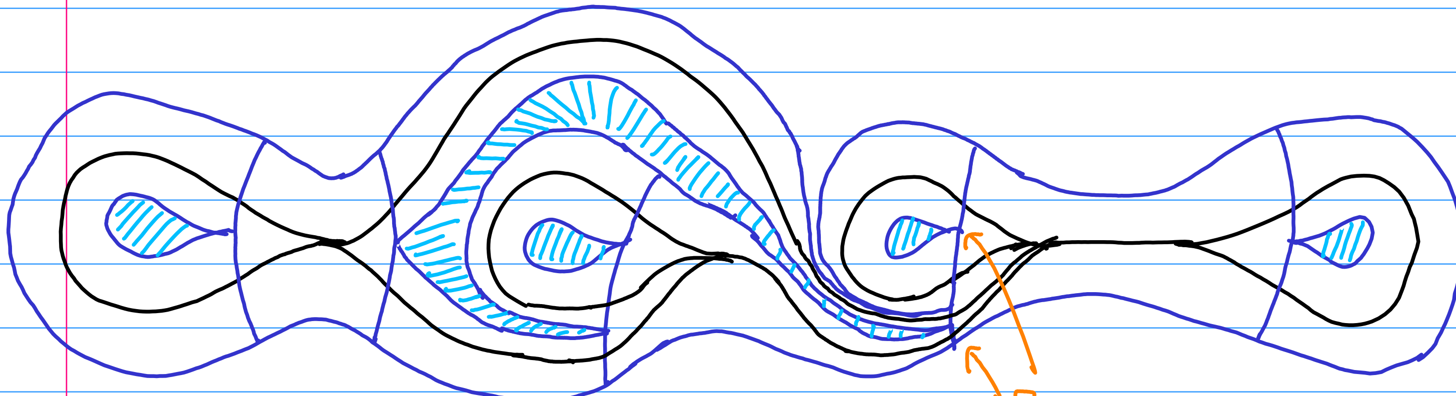
$\downarrow \sigma_1\sigma_2$ Skip over the 2 & 3.



$\downarrow \sigma_3^{-1}$



In terms of a fibrated neighbourhood:



7 cusps in all

We can fit the "image" train track inside of the fibrated neighbourhood.

Think of this as carrying the transformed measure!

Train track map:

$a \mapsto \bar{d}\,\bar{c}\,\bar{2}$

$b \mapsto \bar{d}\,\bar{a}\,\bar{1}$

$c \mapsto b$

$d \mapsto c\,d\,\bar{4}\,\bar{d}$

$1 \mapsto 4$

$2 \mapsto 1$

$3 \mapsto 2$

$4 \mapsto 3$

Incidence matrix:

$$M = \begin{array}{cc} \begin{array}{cccc} a & b & c & d \end{array} & \\ \begin{pmatrix} 0 & 0 & 1 & 1 \\ 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 2 \end{pmatrix} & \begin{array}{c} f(a) \\ f(b) \\ f(c) \\ f(d) \end{array} \end{array}$$

Can we tell from the incidence matrix if this is a pseudo-Anosov?

We can, as long as the train track map is efficient (later).

$$M^2 = \begin{pmatrix} 0 & 1 & 1 & 2 \\ 0 & 0 & 2 & 3 \\ 1 & 0 & 0 & 1 \\ 0 & 1 & 2 & 4 \end{pmatrix}, \quad M^3 = \begin{pmatrix} 1 & 1 & 2 & 5 \\ 0 & 2 & 3 & 6 \\ 0 & 0 & 2 & 3 \\ 1 & 2 & 4 & 9 \end{pmatrix}, \quad M^4 = \begin{pmatrix} 1 & 2 & 6 & 12 \\ 2 & 3 & 6 & 14 \\ \boxed{0} & 2 & 3 & 6 \\ 2 & 4 & 10 & 21 \end{pmatrix}$$

$$M^5 = \begin{pmatrix} 2 & 6 & 13 & 27 \\ 3 & 6 & 16 & 33 \\ 2 & 3 & 6 & 14 \\ 4 & 10 & 23 & 48 \end{pmatrix}$$

No more zeros!

A matrix A is <u>reducible</u> if there exists a permutation matrix P such that P^TAP is block-triangular. An <u>irreducible</u> matrix is not reducible. (d'uh)

Equivalently: A is irreducible if, $\forall i,j$, there exists k such that

$$(A^k)_{ij} > 0.$$

Hence, the matrix M above is irreducible.

<u>Perron-Frobenius theorem:</u> Let $A = (a_{ij})$ be a real $n \times n$ matrix with $a_{ij} \geq 0$. Then:

1. The largest eigenvalue of A is the spectral radius, λ. (i.e., the spectral radius is real)

2. The corresponding eigenvector has nonnegative entries.

3. $\min_i \sum_j a_{ij} \leq \lambda \leq \max_i \sum_j a_{ij}$

[The theorem is slightly different for $a_{ij} > 0$. The the largest eigenvalue is nondegenerate, and the eigenvector has entries > 0.]

3. implies that if we have an irreducible matrix with $\lambda = 1$, then it must be a permutation matrix. (apply 3 to A^n, for n large enough.)

For the matrix M above, $\lambda \simeq 2.2966 > 1$, so it is indeed pseudo-Anosov.