Braids Lecture 35: Train Track Automata

So far the train-tracks associated with pA have appeared from nowhere: we simply verified that a given train track "supported" a pA.

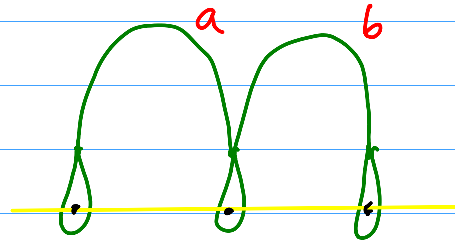
Two natural questions now arise:

1. Given a braid associated with an isotopy class, is there an algorithm that yields a train track supporting the pA? [yes — Bestvina-Handel, say]

2. Is there a way to "enumerate" pA's using train tracks?

We look at 2 first, for the next few lectures.

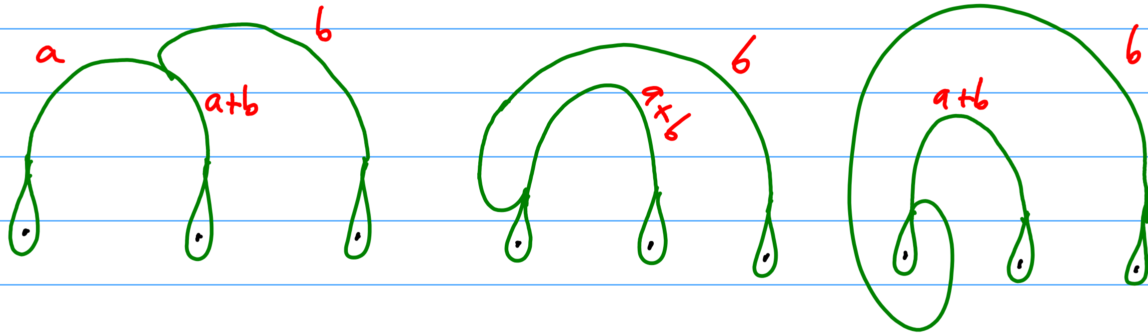
Consider a measured train track for 3 punctures in normal form:



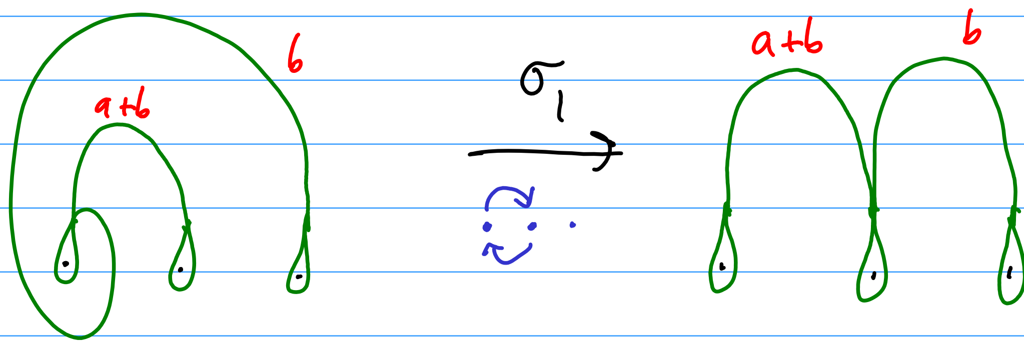
a, b is the measure of each real edge

We say that the track is properly embedded if, after aligning the punctures on a horizontal line, the only segments of train track below the line are infinitesimal.

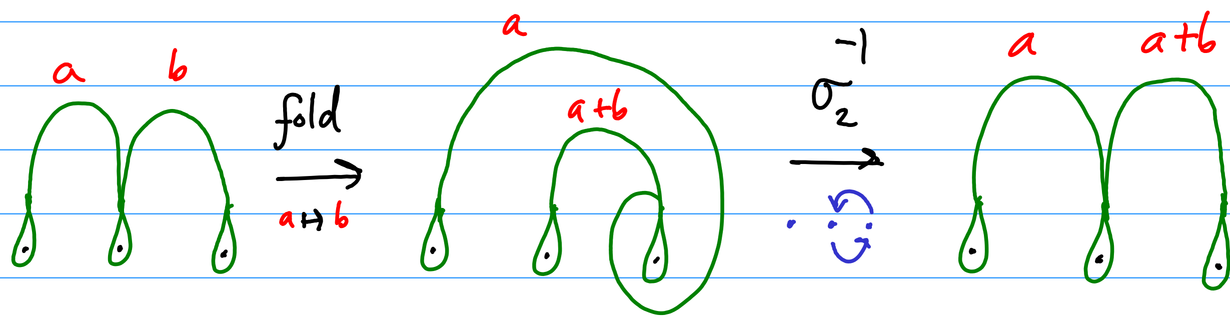
Now recall the "elementary foldings" from before:



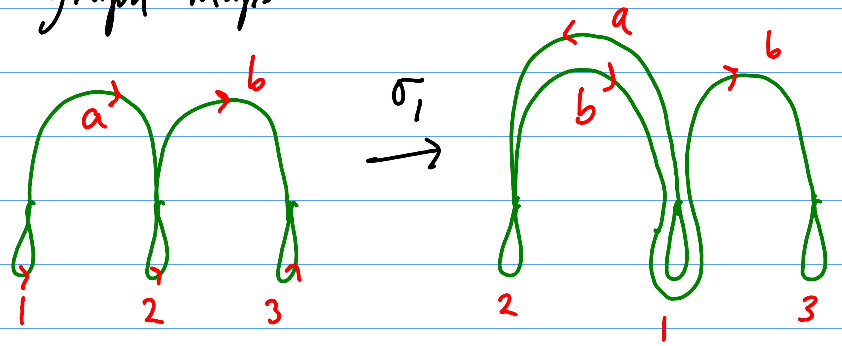
We have folded edge b onto a, folding past the infinitesimal loop. The new track is not properly embedded, but it can be made so after jumping puncture 1 over 2:



This is exactly the same graph as we had originally.
There is only one possible other folding: a onto b:



Here is another way to regard the first folding, as a graph map:

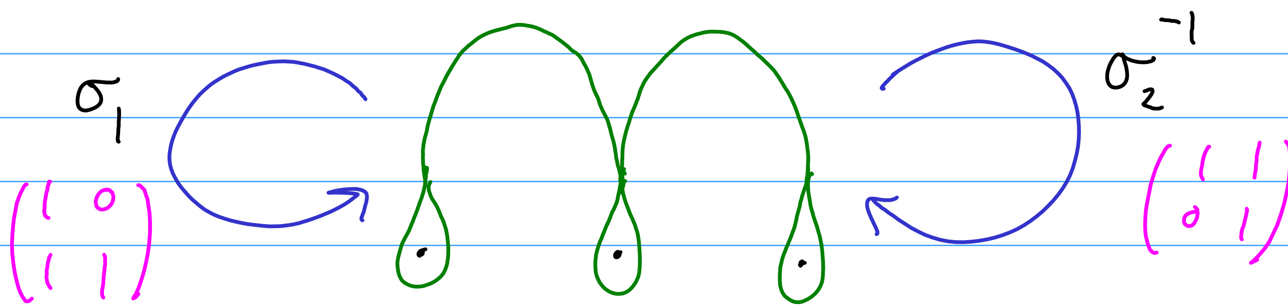


$a \mapsto \bar{a}$
$b \mapsto a2b$
$1 \mapsto 2$
$2 \mapsto 1$
$3 \mapsto 3$

Hence, the incidence matrix is $\begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}$.

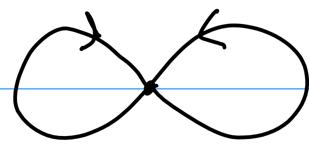
For the second folding it is $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$.

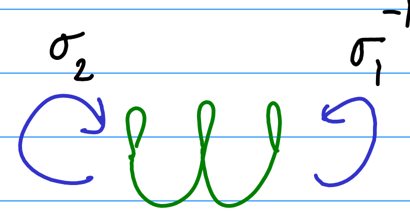
We have just constructed our first <u>train track automaton</u>:



What does this mean? A train track automaton is a graph, consisting of vertices and directed edges. To each edge is associated a nonnegative matrix and a braid word. Each vertex is a train track — an automaton is a graph of graphs!

↑
properly embedded, normal

Simpler form: 

There is an equivalent "mirror image": 

<u>Claim:</u> we can generate all possible pA's with a given singularity data by examining closed paths in the automaton.

<u>Procedure:</u>

1. Choose any vertex as the "starting vertex", set $M = I$, $\gamma = e$.
2. Follow an edge in the automaton with incidence matrix $M^{(i)}$ and braid $\beta^{(i)}$. Update $M = M \cdot M^{(i)}$, $\gamma = \gamma \cdot \beta^{(i)}$
3. Repeat 2 until we are back at the starting vertex, not necessarily for the first time. If M is irreducible, then the braid γ labels a pA isotopy class with the same singularity data as the vertices of the automaton, and M is its incidence matrix.

<u>Example:</u> In the graph above, follow the left arrow, followed by the right:

$$\gamma = \sigma_1 \cdot \sigma_2^{-1}, \qquad M = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} \cdot \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix}.$$

In fact, since σ_1^m has matrix $\begin{pmatrix} 1 & 0 \\ m & 1 \end{pmatrix}$, $m \geq 0$

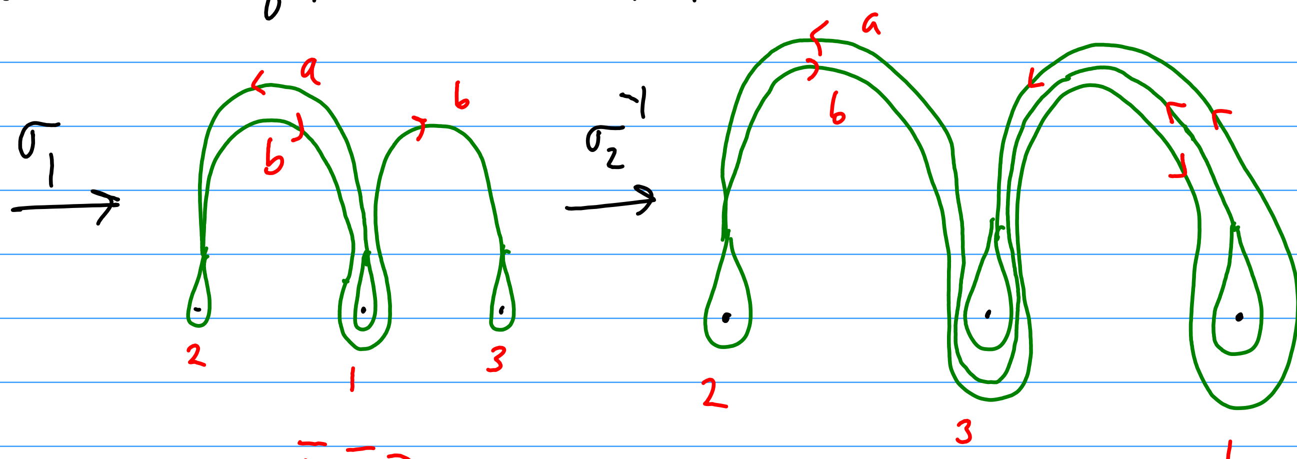
σ_2^{-n} " " $\begin{pmatrix} 1 & n \\ 0 & 1 \end{pmatrix}$ $n \geq 0$

then clearly $\sigma_1^m \sigma_2^{-n} = \begin{pmatrix} 1 & n \\ m & 1+mn \end{pmatrix}$ is pA for $m > 0$ & $n > 0$.

So any combination of σ_1 and σ_2^{-1} is pA!

Step 3 is the invariance property of the train track, and hence of the measured foliation: returning to the starting vertex "closes the loop" and gives us a candidate for a pA.

Quick check of the incidence matrix:



Map: $a \mapsto \bar{b}\, 2\, \bar{a}$
$b \mapsto a\, 2\, b\, 3\, \bar{b}$

$$M = \underset{\sigma_1}{\begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}} \underset{\sigma_2^{-1}}{\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}} = \begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix}$$