Lecture 1: Stirring & Mixing

Stirring: mechanical action
Mixing: homogenization of a scalar

\[ \theta(x, t) = \text{concentration}, \quad u(x, t) \text{ given} \]

**Advection-Diffusion Eq:**

\[ \frac{\partial \theta}{\partial t} + u \cdot \nabla \theta = \kappa \nabla^2 \theta \quad \nabla \cdot u = 0 \quad \text{in } \Omega \]

Boundary conditions:

\[ \hat{n} \cdot \nabla \theta = 0 \quad \text{on boundary } \partial \Omega \]

\[ \hat{n} \cdot u = 0 \]

Let \( \langle \cdot \rangle = \int_{\Omega} \cdot \ dV \)

Multiply AD by \( \theta^{m-2} \), integrate:

\[ \langle \theta^{m-1} \partial_t \theta \rangle = \partial_t \langle \theta^m \rangle \]

\[ \langle \theta^{m-1} u \cdot \nabla \theta \rangle = \langle u \cdot \nabla \theta^m \rangle = \langle \nabla \cdot (u \theta^m) \rangle \]

\[ = \int_{\partial \Omega} \theta^m u \cdot \hat{n} \ dS = 0 \]

\[ \langle \theta^{m-1} \nabla^2 \theta \rangle = \kappa \int_{\Omega} \langle \nabla \cdot (\theta^{m-1} \nabla \theta) - \nabla \theta^{m-1} \cdot \nabla \theta \rangle \]

\[ = \kappa \int_{\partial \Omega} \theta^{m-1} \nabla \theta \cdot \hat{n} \ dS - \kappa m (m-1) \langle \theta^{m-2} |\nabla \theta|^2 \rangle \]
\[
\partial_t \langle \theta^m \rangle = -\kappa m (m-1) \langle \theta^{m-2} \rangle \nabla \theta^2
\]

\[m=0 \text{ is trivial}\]

\[m=1: \partial_t \langle \theta \rangle = 0 \quad \text{Total amount of } \theta \text{ is conserved}\]

\[m=2: \partial_t \langle \theta^2 \rangle = -2\kappa \langle \nabla \theta^2 \rangle \quad \langle \theta^2 \rangle \text{ non-increasing!}\]

Let variance \( \text{Var} = C_2 = \langle \theta^2 \rangle - \langle \theta \rangle^2 \)

\[\partial_t C_2 = -2\kappa \langle \nabla \theta^2 \rangle \quad \text{constant}\]

**Scenario:** \( C_2 \)

- Variance can only decrease.
- Slows down as \( \langle \nabla \theta^2 \rangle \to 0 \)
- But \( \langle \nabla \theta^2 \rangle = 0 \) iff \( \theta = \text{constant} \)

Hence the system is "driven" towards a homogeneous state where

\[\theta(x,t) = \langle \theta \rangle = \text{constant.} \quad (C_2 = 0, \langle \theta^2 \rangle = \langle \theta \rangle^2)\]

No fluctuations from the mean! When \( C_2 \) is small "enough", we say the system is mixed.

**Big Q:** Where is \( y(x,t) \) ? (stirring)

It doesn't appear in the variance equation!
But of course the variance equation is not closed: it depends on $\nabla \theta$.

What happens when you stir?

A "blob" (Gaussian patch, say) evolves into a filamentation and "striations".

This hints at the answer: *stirring increases* $\nabla \theta$

$$\frac{\partial}{\partial t} \langle \theta^2 \rangle = -2\kappa \langle |\nabla \theta|^2 \rangle$$

This becomes larger as we stir.

By how much are gradients increased? After all, if $|\nabla \theta|$ becomes too large, then $\langle \theta^2 \rangle \to 0$, so there are no gradients anymore.

*Answer*: for "good" stirring, the system is driven to a state where

$$\kappa \langle |\nabla \theta|^2 \rangle \to \text{independent of } \kappa$$

Hence,

$$\nabla \theta \sim \kappa^{-1/2}$$

This is the chaotic/turbulent mixing scenario:

$$\frac{\partial}{\partial t} \langle \theta^2 \rangle$$

becomes independent of $\kappa$ after a "short" transient.

(How short? Typically $\sim \log \kappa$)
Furthermore, the smallest scales visible in the concentration field \( \theta(x, t) \) have size \( \sim \sqrt{\kappa t} \). (missing a dimensional factor \( \rightarrow \) see later)

Note that \( \mathbb{E}[\theta^2] \) independent of \( \kappa \) is crucial; in most applications, \( \kappa \) is tiny!

Heat: \( \kappa = 2.2160 \times 10^{-5} \text{ m}^2/\text{s} \) at 300 K

10 m room: diffusion time \( \sim \frac{L^2}{\kappa} = (10 \text{ m})^2 \sim 4.5 \times 10^6 \text{ sec} \)
\[
\frac{L^2}{(2 \times 10^{-5} \text{ m}^2/\text{s})} \sim 1300 \text{ hours} \approx 53 \text{ days}!
\]

So we better stir! Even thermal convection is often enough.

Example of a good mixer:

\[ \frac{\partial}{\partial t} \theta = (\lambda x - \lambda y) \]

"hyperbolic point"

\[ \kappa \nabla^2 \theta \]

Can solve this exactly (we'll say more next time), but let's do the simplest thing: look for an \( x \)-independent solution of the form:

\[ \theta(x, t) = e^{-\lambda t} f(y) \]

\[ -\lambda f - \lambda y f' = \kappa f'' \]

Boundary condition:

\( f \to 0 \) as \( y \to \pm \infty \).
Solution is: \( f(y) = e^{-y^2/2\ell^2} \), where \( \ell^2 = \frac{\kappa}{\lambda} \)

Hence, \( \Theta(x, t) \sim e^{-2\pi t} e^{-y^2/2\ell^2} \)

This is the "filament" solution:

\[
\begin{array}{c}
\text{Cross-section is} \\
\text{Gaussian with} \ \ell \\
\text{no structure in} \ x.
\end{array}
\]

In fact, this solution tells us about the ultimate state of any compactly-supported initial condition:

"blob" \[\rightarrow\] "filament"

\( \text{central part} \sim \text{Gaussian cross-section} \)

\( \text{intensity} \sim e^{-2\pi t} \)

For this case, we know the length scale \( \ell \) "strictures":

\[
\ell = \sqrt{\frac{\kappa}{\lambda}} \quad \text{Batchelor length}
\]

Note \( \ell \sim \sqrt{\kappa} \), as necessary to make decay rate independent of \( \kappa \).

In practical applications, \( \ell \) is often taken to be the local rate of strain.
$l$ is set by a balance between compression and diffusion

\[ \text{(diffuse)} \rightarrow l \rightarrow \text{(compress)} \]

Summary: how mixing proceeds

- A blob is stirred →

- For a while, $\langle \Theta^2 \rangle$ is $\sim$ constant, since $\kappa$ is small

- When $\nabla \Theta$ reaches scales of order $l$, diffusion takes over

- After that, $\langle \Theta^2 \rangle$ decays at a $\kappa$-independent rate

\[ \langle \Theta^2 \rangle \]

\[ T \text{ given by: } \frac{1}{2} T \sim \sqrt{\kappa} \]

\[ T \sim \lambda^{-1} \log_{10} \kappa \]

\[ T \]

\[ t \]

filamentation phase → mixing phase
Effective Diffusivity

\[ l \sim \sqrt{K/\lambda} \]

Recall: filaments in chaotic advection

Goal was to compute decay of variance,

\[ \langle \theta^2 \rangle \sim e^{-\sigma t} \quad (\sigma = \lambda \text{ for uniform strain}) \]

But when can we replace the advection-diffusion equation by an "effective" diffusion equation?

\[ \frac{\partial \theta}{\partial t} + \mathbf{u} \cdot \nabla \theta = \kappa \nabla^2 \theta \quad \Rightarrow \quad \frac{\partial \theta}{\partial t} = K_{\text{eff}} \nabla^2 \theta ? \]

Diffusion arises from noise: \( x_n = x_{n-1} + \xi_n \)

Assume \( \langle \xi_n \rangle = 0 \), \( \langle \xi_n^2 \rangle = \sigma^2 \) i.i.d.

\[ x_n = x_0 + \sum_{\lambda=1}^{n} \xi_n \quad \langle x_n \rangle = 0 \]

"time"

\[ \langle x_n^2 \rangle = \sum_{\lambda=1}^{n} \langle \xi_n^2 \rangle = n \sigma^2 = 2Kt \]

In \( d \) dimensions,

\[ \langle x_n^2 + y_n^2 (+ z_n) \rangle = n d \sigma^2 = 2dKt\]

\[ K_{\text{eff}} = \frac{\sigma^2}{2T} \]
Now if we take a "cloud" of points 🌎, and define a density

\[ \theta(x, t) = \text{density of points} \]

Then \( \theta \) satisfies

\[ \frac{\partial \theta}{\partial t} = \kappa \nabla^2 \theta \]

if each point evolves independently according to

\[ x' = x + \xi. \]

Of course, this requires "coarse-graining": it is only true if we don’t look to closely (scale \( \lesssim \sigma \)) or too often (time scale \( \lesssim T \)).

This provides clues as to when the concept of an effective diffusivity makes sense.

Rest of lecture: look at an example, the famous sine flow.

- Velocity field (shear flow)

\[ \begin{align*}
\mathbf{u} &= \left( U \sin\left(\frac{2\pi ky}{L}\right), 0 \right),
\quad \text{applied for } 0 \leq t < \tau/2 \\
\mathbf{v} &= \left( 0, U \sin\left(\frac{2\pi kx}{L}\right) \right),
\quad \text{for } \tau/2 \leq t < \tau.
\end{align*} \]

Can solve \( \dot{x} = \mathbf{u} \), \( x(0) = x_0 \) exactly:

**Step 1:**

\[ x(\tau/2) = x_0 + Ut/2 \sin\left(\frac{2\pi ky_0}{L}\right) \]

\[ y(\tau/2) = y_0. \]
**STEP 2:**
\[ x(t) = x(t/2) \]
\[ y(t) = y(t/2) + \frac{U t}{2} \sin \left( \frac{2\pi k x(t/2)}{L} \right) \]

Write as one map of period \( T \):
\[
x' = x + T \sin \left( \frac{2\pi k y}{L} \right) \quad T = \frac{U t}{2}
\]
\[
y' = y + T \sin \left( \frac{2\pi k x'}{L} \right)
\]

Easy to iterate on a gazillion particles. Note \( x' \)!

**Example 1:** Run Matlab script example (1).

\[ L = \mathcal{L} = 1, \quad T = 0.1 \]

Note how regular the orbits are: for small \( T \) the map is effectively a symplectic integrator

\[
\frac{x' - x}{T} = \sin \left( \frac{2\pi k y}{L} \right), \quad \frac{y' - y}{T} = \sin \left( \frac{2\pi k x'}{L} \right)
\]

As \( T \to 0 \), this approximate

\[ \frac{dx}{dt} = \sin \left( \frac{2\pi k y}{L} \right), \quad \frac{dy}{dt} = \sin \left( \frac{2\pi k x}{L} \right), \]

or flow with stream function:
\[
\psi = \frac{L}{2\pi k} \left( \cos \left( \frac{2\pi k x}{L} \right) - \cos \left( \frac{2\pi k y}{L} \right) \right)
\]
The streamline aren't traced exactly because $T$ is finite.

Example 2 adds a bit of noise.

\[ x' = (\text{sine map}) + \sqrt{2D} \Xi \]

Example 3: $T = 1$. Now doesn't approximate a flow at all \textbf{chaotic}.

Example 4: $T = 1$, $L = 1$, $D = 10^{-4}$: "fat" filaments.

\rightarrow measure width by clicking

\rightarrow repeat for $D = 10^{-6}$

\rightarrow observe rough $\sqrt{D}$ scaling for filament width

(see Lecture 1)

Example 5: $T = \frac{1}{2}$, $L = 1$, $D = 10^{-6}$, make $L$ larger.

Plot $\langle x^2 \rangle$ vs iteration $\langle x \rangle$.

Hence, the concept of an effective diffusivity makes sense if we look at large scales such that we cannot see the correlated small scale motions, and long times.

(but not too long!)

\rightarrow Useful for turbulence $K_{eff} \approx 0.068 \Rightarrow D = 10^{-6}$

Note that the "cross" shape evident in the pattern is not captured.
Lecture 2: Stirring by swimming organisms

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A controversial proposition:

- There are many regions of the ocean that are relatively quiescent, especially in the depths (1 hairdryer/ km\(^3\));
- Yet mixing occurs: nutrients eventually get dredged up to the surface somehow;
- What if organisms swimming through the ocean made a significant contribution to this?
- There could be a local impact, especially with respect to feeding and schooling;
- Also relevant in suspensions of microorganisms (Viscous Stokes regime).
Bioturbation

The earliest case studied of animals ‘stirring’ their environment is the subject of Darwin’s last book.

This was suggested by his uncle and future father-in-law Josiah Wedgwood II, son of the famous potter.

“I was thus led to conclude that all the vegetable mould over the whole country has passed many times through, and will again pass many times through, the intestinal canals of worms.”
Munk’s Idea

Though it had been mentioned earlier, the first to seriously consider the role of ocean biomixing was Walter Munk (1966):

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Abyssal recipes

WALTER H. MUNK*

(Received 31 January 1966)

Abstract—Vertical distributions in the interior Pacific (excluding the top and bottom kilometer) are not inconsistent with a simple model involving a constant upward vertical velocity \( w \approx 1 \cdot 2 \text{ cm day}^{-1} \) and eddy diffusivity \( \kappa \approx 1 \cdot 3 \text{ cm}^2 \text{ sec}^{-1} \). Thus temperature and salinity can be fitted by exponential-like solutions to \( \left[ \kappa \cdot d^2/dz^2 - w \cdot d/dz \right] T, S = 0 \), with \( \kappa/w \approx 1 \text{ km} \) the appropriate “scale height.” For Carbon 14 a decay term must be included, \( [ \ ]^{14C} = \mu^{14C} \); a fitting of the solution to the observed \(^{14}\text{C}\) distribution yields \( \kappa/w^2 \approx 200 \text{ years} \) for the appropriate “scale time,” and permits \( w \) and
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“...I have attempted, without much success, to interpret [the eddy diffusivity] from a variety of viewpoints: from mixing along the ocean boundaries, from thermodynamic and biological processes, and from internal tides.”
Basic claims

The idea lay dormant for almost 40 years; then

- Huntley & Zhou (2004) analyzed the swimming of 100 (!) species, ranging from bacteria to blue whales. Turbulent energy production is $\sim 10^{-5} \text{ W kg}^{-1}$ for 11 representative species.

- Total is comparable to energy dissipation by major storms.

- Another estimate comes from the solar energy captured: 63 TeraW, something like 1% of which ends up as mechanical energy (Dewar et al., 2006).

- Kunze et al. (2006) find that turbulence levels during the day in an inlet were 2 to 3 orders of magnitude greater than at night, due to swimming krill.
In situ experiments

Katija & Dabiri (2009) looked at jellyfish:

[movie 1] (Palau’s Jellyfish Lake.)
Displacement by a moving body

Maxwell (1869); Darwin (1953); Eames et al. (1994); Eames & Bush (1999)
Cylinders and spheres: Displacements

\[ \Delta_L^2(a, b) a \text{ (cylinder)} \]

\[ \Delta_L^2(a, b) a^2 \text{ (sphere)} \]
Displacement for cylinders

Small $a$: $\Delta \sim -\log a$
Large $a$: $\Delta \sim a^{-3}$
(Darwin, 1953)

\[
\int_0^1 \Delta^2(a) da \simeq 2.31
\]
\[
\int_1^\infty \Delta^2(a) da \simeq .06
\]

$\implies$ 97% dominated by “head-on” collisions (similar for spheres)
Numerical simulation

- Validate theory using simple simulations;
- Large periodic box;
- $N$ swimmers (cylinders of radius 1), initially at random positions, swimming in random direction with constant speed $U = 1$;
- Target particle initially at origin advected by the swimmers;
- Since dilute, superimpose velocities;
- Integrate for some time, compute $|x(t)|^2$, repeat for a large number $N_{\text{real}}$ of realizations, and average.
A ‘gas’ of swimmers

[movie 2] \( N = 100 \) cylinders, box size = 1000
How well does the dilute theory work?

\[
\frac{\langle |x|^2 \rangle}{2nU^3} \text{ theory}
\]

- \( n = 10^{-3} \)
- \( n = 5 \times 10^{-4} \)
- \( n = 10^{-4} \)
Cloud of particles

[movie 3] (30 cylinders)
Cloud dispersion proceeds by steps

\[ \langle |x|^2 \rangle \]

- \( N = 30 \)
- \( n = 7.5 \times 10^{-4} \)
Considerable literature on transport due to microorganisms: Wu & Libchaber (2000); Hernandez-Ortiz et al. (2006); Saintillian & Shelley (2007); Underhill et al. (2008); Ishikawa (2009); Leptos et al. (2009)

Lighthill (1952), Blake (1971), and more recently Ishikawa et al. (2006) have considered squirmers:

- Sphere in Stokes flow;
- Steady velocity specified at surface, to mimic cilia;
- Steady swimming condition imposed (no net force on fluid).

(Drescher et al., 2009)  (Ishikawa et al., 2006)
Typical squirmer

3D axisymmetric streamfunction for a typical squirmer, in cylindrical coordinates \((\rho, z)\):

\[
\psi = -\frac{1}{2} \rho^2 + \frac{1}{2r^3} \rho^2 + \frac{3\beta}{4r^3} \rho^2 z \left( \frac{1}{r^2} - 1 \right)
\]

where \(r = \sqrt{\rho^2 + z^2}\), \(U = 1\), radius of squirmer = 1.

Note that \(\beta = 0\) is the sphere in potential flow.

We will use \(\beta = 5\) for most of the remainder.
Particle motion for squirmer

A particle near the squirmer’s swimming axis initially (blue) moves towards the squirmer.

After the squirmer has passed the particle follows in the squirmer’s wake.

(The squirmer moves from bottom to top.)

[movie 4]
Squirmer displacements
Squirmers: Transport

\[ \langle |x|^2 \rangle \]
Squirmers: Trajectories

$b/\lambda = 0$

$b/\lambda = 0.5$

$b/\lambda = 1$
Far field: Displacements
Far field: transport

\[ \frac{\kappa}{U\ell^4} \]

\[ \beta \]

\[ 10^{-1} \]

\[ 10^0 \]

\[ 10^1 \]

\[ 10^2 \]

\[ 10^3 \]

\[ 10^{-2} \]

\[ 10^{-1} \]

\[ 10^0 \]

\[ 10^1 \]

\[ 10^2 \]

\[ 10^3 \]
Finite Reynolds number: Displacements

\[ \frac{b}{\lambda} \log \left( \frac{a}{\ell} \right) \]

\[ -0.5 \quad 0 \quad 0.5 \quad 1 \quad 1.5 \]

\[ -2 \quad -1 \quad 0 \quad 1 \quad 2 \quad 3 \quad 4 \]

\[ 0 \quad 1 \quad 2 \quad 3 \quad 4 \quad 5 \quad 6 \quad 7 \quad 8 \quad 9 \quad 10 \]
Finite Reynolds number: Transport

\[ \kappa / U n \ell^4 = 5.9 \, R e^{-0.61} \]
Lecture 2: Stirring by swimming organisms

Target particle moves under the influence of many swimmers

Single swimmer: take a cylinder

How do we compute \( \Delta \)?

- Swimming velocity is \( V \) (amr)
- Straight line for distance \( \lambda \)
- Axially-symmetric, steady swimmer
- \( a, b \) are "impact parameters" \((a>0)\)

Compute \( \Delta_{\lambda}(a, b) \)
Do 2D case (axisymmetric 3D similar):

\[ x, y, u(x, y) \]

Comoving Frame (steady u)

\[ (x, y, u) \]

\[ Ax = (b, y) \]

\[ \Delta x = \Delta x \]

\[ x_f = b - \lambda + \Delta x \]

\[ d\tilde{x} = \tilde{u}(\tilde{x}(t) + b - Ut, \tilde{y}(t) + \lambda) \]

\[ \frac{dx}{dt} + U = \tilde{u}(x, y) \implies \frac{dx}{dt} = -U + \tilde{u}(x, y) = u(x, y) \]

\[ -\lambda + \Delta x = \int_0^T \frac{\Delta x}{u(x, y)} \, dt \]

\[ \lambda = \int_0^b \frac{dx}{u} = -\int_b^{b+\Delta x} \frac{dx}{u} = \int_{b+\Delta x}^{b+\lambda + \Delta x} \frac{dx}{u} \]

\[ T = \frac{T}{U} = \int_0^{x_f} \frac{dx}{u(x, y)} \]

Alternate form: \( T = \frac{\lambda}{U} = \int_b^{b+\lambda + \Delta x} \frac{dx}{u} \)

\[ x_f = b - \lambda + \Delta x \]

Moving at speed \( U \)

\[ \text{position} = (b - Ut, y) \]

For \( y \):

\[ \frac{dy}{dt} = \tilde{u}(x, y) \]

For \( x \):

\[ \frac{dx}{dt} = -U + \tilde{u}(x, y) = u(x, y) \]

\[ \text{autonomous (better!)} \]

\[ \text{If particle doesn't move much and } |b-1| \text{ "large", then } |u| \leq U \]
\[
\frac{\lambda}{U} \sim \int_{b-\Delta x}^{b} \frac{dx}{|u|} - \frac{\Delta x}{U} \Rightarrow \Delta x = \int_{b-\Delta x}^{b} \frac{dx}{|u|} - \frac{\lambda}{U}
\]

\[
\Delta x \sim \int_{b-\Delta x}^{b} \left( \frac{1}{|u|} - \frac{1}{U} \right) dx
\]

"Rayleigh form"

Better form, since now can take \( b \to \infty \), \( b-\Delta x \to -\infty \)

if we went

Intuitively, this formula measures the "lay" behind a free-streaming particle:

Swimmer:

\[ t = T \quad t = 0 \]

Free:

\[ t = T \quad t = 0 \]

\[ T = \frac{\lambda}{U} \]

2D incompressible:

\[ u = \frac{\partial \Psi}{\partial y}, \quad v = -\frac{\partial \Psi}{\partial x} \]

\[ \Psi(x_f, a + \Delta y) = \Psi(b, a) \quad \text{Same streamline} \]

\[ \Psi(b-\lambda + \Delta x, a + \Delta y) = \Psi(b, a) \quad \text{solve for } \Delta y, \text{ given } \Delta x \]

If \( |b-\lambda| \gg \Delta x \),

\[ \Psi(b-\lambda, a + \Delta y) \approx \Psi(b, a) \quad \text{solve for } \Delta y \]

If also \( \Delta y \ll a \),

\[ \Psi(b-\lambda, a) + \partial y \partial y \Psi(b-\lambda, a) \simeq \Psi(b, a) \]

\[ \frac{\Delta y}{u(b-\lambda, a)} \]

\[ \Delta y \simeq \frac{\Psi(b, a) - \Psi(b-\lambda, a)}{u(b-\lambda, a)} \]
Now for infinitesimal $\lambda$, we have:

$$
\Delta y = \frac{\psi(\infty, a) - \psi(-\infty, a)}{\lambda} = 0
$$

$$
\Delta y = 0
$$

for $\lambda \to \infty$, $b-1 \to -\infty$

Cylinder in potential flow:

$$
\psi(x,y) = -Uy \left(1 - \frac{\lambda^2}{x^2+y^2}\right)
$$

Set $U = \lambda = 1$

Far away, $\psi \sim \frac{y}{r^2}$, so $\tilde{u} \sim \frac{1}{r^2}$

However, trajectories are almost closed.

Net result is $\Delta(y) \sim \frac{1}{a^3}$ Much smaller than overall excursion!

The limit $a \ll 1$ is more interesting.

$\psi \sim -2(r-1)\cos \theta$ (as $\theta \to \infty$)

$\psi \sim -(1 - \frac{1}{a^2})y$ (as $\theta \to 0$)

Stagnation point $\psi \sim -2(x-1)y$ (as $\theta \to 0$)
Need to calculate \( \int \left( \frac{1}{u} + 1 \right) dx \) over each region 1, 2, 3.

**Region 1:** \( \Psi = \Psi(b, a) = -(1 - b^{-2})a \)

\[
u = -(1 - x^{-2}), \quad T_1 = \int_b^l \left( \frac{1}{u} + 1 \right) dx = \int_b^{1+\epsilon} \frac{dx}{1 - x^2}
\]

*Transit time*

After using \( \epsilon \ll 1, \ b \gg 1 \):

\[
T_1 \approx \frac{1}{2} \log \left( \frac{2}{\epsilon} \right) + \epsilon^2 - b^{-1} + O(\epsilon^3, b^{-2})
\]

**Region 2:** \( (x_1, y_1) \)

\[
X = x - 1, \quad Y = y
\]

\[
\Psi = -2X Y
\]

At \( x_0, y_0 \):

\[
\Psi = -2x_0 y_0 = -(1 - b^{-2})a
\]

\[\Rightarrow y_0 = \frac{a}{2 \epsilon}\]

But also \( y_1 = \epsilon \), so \( x_1 = \frac{9}{2 \epsilon} \).

\[
T_2 = \int_{x_0}^{x_1} \left( \frac{1}{u} + 1 \right) dx = \int_{\epsilon}^{\frac{9}{2 \epsilon}} \left( \frac{1}{u} + 1 \right) dx = -\frac{1}{2} \log \left( \frac{2 \epsilon^2}{9} \right) + \frac{\epsilon}{2 \epsilon} - \epsilon
\]
Regime 3:

\[ T_3 = \int_{\theta_1}^{\pi/2} \left( \frac{1}{u} + 1 \right) \frac{dx}{d\theta} \quad \text{for small } \theta_1 \]

\[ = \frac{1}{2} \int_{\theta_1}^{\pi/2} \frac{\cos 2\theta}{\sin \theta} \, d\theta \]

\[ \approx -1 + \frac{1}{2} \log 2 - \frac{1}{2} \log \theta_1 + O(\theta_1^2) \]

\[ \tan \theta_1 = \frac{Y_1}{1+X_1} = \frac{\epsilon}{1+\epsilon} \approx \epsilon (1-\frac{\epsilon}{2}) = \epsilon - \frac{\epsilon}{2} \]

\[ \therefore T_3 \approx -1 + \frac{1}{2} \log 2 - \frac{1}{2} \log \epsilon + \frac{1}{2} \frac{\epsilon}{2} + O(\epsilon^2) \]

Add everything together:

\[ T = T_1 + T_2 + T_3 = \left( \frac{1}{2} \log \left( \frac{2}{\epsilon^2} \right) - 6^{-1} \right) + \left( -\frac{1}{2} \log \left( \frac{2\epsilon}{\epsilon^2} \right) \right) \]

\[ + \left( -1 + \frac{1}{2} \log 2 - \frac{1}{2} \log \epsilon \right) \]

\[ T = -\frac{1}{2} \log \alpha - 1 + \frac{3}{2} \log 2 - 6^{-1} \quad \text{to leading order,} \]

\[ \text{Dominant term for small } \alpha. \]

\[ \text{comes only from region 2, near stagnation point.} \]

\[ T \to \infty \text{ for } \alpha \to 0, \quad \text{particle gets stuck!} \]

The total drift is given by \( 2T \), since the body is fore-aft symmetric.

In general, the coefficient of \( \log \alpha \) is given by summing over the linearization calls for each (hyperbolic) stagnation pt encountered. (not true for no-slip!)
Note that to pick up the contribution, the target particle must come in the vicinity of the stagnation point:

\[ \Delta \chi(a,b) = \begin{cases} -\log a, & 0 \leq b \leq 1 \\ \neg \text{(neglect)}, & \text{otherwise} \end{cases} \]

Effective diffusivity:

What we have: \( \Delta \chi(a,b) \) \quad \text{Need: effective diffusivity}

Constants: \( U, \lambda, l, n \) \quad \text{Random: } a, b

If we pick a random point in space, what is PDF of \( a, b \)?

\[ \frac{1}{V} \, dx \, dy \rightarrow \rho(a,b) \, da \, db \]

Assume target particle at origin:

- Hard way: compute \((a,b)\) from \((x,y)\), transform variables.
- Easier: note \((a,b)\) just like \((x,y)\), but rotated, and \( a > 0 \).

Hence:

\[ \frac{1}{V} \, dx \, dy = \frac{1}{V} \, 2 \, da \, db \quad 2D \]
In 3D, \[ \frac{1}{V} \int \, dx \, dy \, dz = \frac{1}{V} \int \, 2\pi a \, da \, db \] like cylindrical coordinates, in hybrid over \( \theta \).

Now, assume target particle is "kicked" by swimmer:

\[ x_N = x_0 + \sum_{k=1}^{N} \Delta_a(a_k, b_k) \hat{r}_k \]

\( a_k, b_k, \hat{r}_k \) random independent identical

On average, particle goes nowhere: \( \langle x_N \rangle = 0 \)

\[ \langle |x_N|^2 \rangle = \sum_{k=1}^{N} \langle \Delta_a^2(a_k, b_k) \hat{r}_k \cdot \hat{r}_k \rangle + \text{vanishing cross terms} \]

\[ = N \langle \Delta_a^2(a, b) \rangle \]

\[ = N \int \Delta_a^2(a, b) \, 2 \, da \, db \] 2D elapsed time

What is \( N \)? # of "collisions" \( N = \frac{t}{T} \) mean free time

\[ T = \frac{1}{\lambda} \] \( \lambda = \text{mean free path} \)

Hence, \( \langle |x(t)|^2 \rangle = \frac{U t}{\lambda} \int \Delta_a^2(a, b) \, 2 \, da \, db \)

Only one swimmer, so \( \int = n \), the number density

\[ \langle |x(t)|^2 \rangle = 2 U n t \int \Delta_a^2(a, b) \, da \, db = 2 d n t = 4 n t \]

What about this?

This depends on integral of squared displacement. Actual mean displaced could be 0!
\[ \kappa = \begin{cases} \frac{U_n}{2 \lambda} \int \Delta^2_{\lambda}(a, b) \, da \, db & \text{2D} \\ \frac{\pi U_n}{3 \lambda} \int \Delta^2_{\lambda}(a, b) \, a \, da \, db & \text{3D} \end{cases} \]

effective diffusivity

Recall our approximate form for cylinder: \( \Delta_{\lambda}(a, b) = \begin{cases} -\log a & 0 < b < \lambda \\ 0, \text{ otherwise} \end{cases} \)

Cylinder: \( \kappa = \frac{2 U_n}{\lambda} \int \frac{\log^2(a)}{a} \, da \)

\[ \int \frac{\log^2 x}{x} \, dx = x \log^2 x - 2x \log x + 2x \]

\[ \int_0^1 \frac{\log^2 x}{x} \, dx = 2 \quad (\text{numerical answer: 2.87}) \]

\[ \kappa \approx U_n \lambda^3 \quad (\text{numerical: } \kappa = 1.19 \, U_n \lambda^3) \]

Note that this is completely independent of \( \lambda \)!

\( \Rightarrow \) see computer simulation.

Another example: consider a swimmer with a bubble "wake":

![Diagram of swimmer with bubble wake]
If a particle is trapped in the bubble, moves by $\Delta x$.

$$\Delta x(a, b) = \int_0^2 a, \text{ particle inside bubble}$$

Total volume of bubble

$$6\pi = 2\pi \int_0^2 \rho \, d\alpha d\theta = \pi \rho \frac{\lambda}{2} V_{\text{bubble}}$$

The $2$ goes away since $2d\alpha d\theta$ is volume element.

$$\kappa = \frac{1}{6} \pi \rho \lambda V_{\text{bubble}}$$

$V_{\text{bubble}}$ = area in 2D, volume in 3D.

Now this depends on path length $\lambda$. This can be much larger than for untrapped fluid. Real swimmer probably in between.

(Viscous swimmer with boundary layer: $\kappa \sim \log \lambda$)

<table>
<thead>
<tr>
<th>Swimmer</th>
<th>$\lambda$ dependence</th>
<th>far/near field dominance</th>
</tr>
</thead>
<tbody>
<tr>
<td>potential</td>
<td>none</td>
<td>near</td>
</tr>
<tr>
<td>(slip)</td>
<td></td>
<td></td>
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<tr>
<td>viscous</td>
<td>none</td>
<td>far</td>
</tr>
<tr>
<td>(slip)</td>
<td></td>
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</tr>
<tr>
<td>squirmer</td>
<td></td>
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</tr>
<tr>
<td>viscous</td>
<td>$\log \lambda$</td>
<td>near</td>
</tr>
<tr>
<td>(no-slip)</td>
<td></td>
<td></td>
</tr>
<tr>
<td>trapped</td>
<td>$\lambda$</td>
<td>near</td>
</tr>
</tbody>
</table>

More topics:
- Green-Kubo
- Walls
- Far field
- Levy flights
- Stratification
GFD Lectures: Swimming & Swirling

Lecture 3: Local Stretching Theories

\[ \partial_t \theta + \mathbf{u} \cdot \nabla \theta = \kappa \nabla^2 \theta. \]

For this lecture, think of \( \theta \) as a "patch."

Last time we examined \( \mathbf{u} = (2x, -2y) \). Let's try something more general:

\[ \mathbf{u} = \mathbf{U} + \mathbf{x} \cdot \mathbf{A}, \quad \nabla \cdot \mathbf{u} = \text{tr} \mathbf{A} = 0. \]

Let \( \langle f \rangle = \int f \, dV \) \( (\mathcal{N} = \mathbb{R}^2 \text{ or } \mathbb{R}^3) \)

Solve (AD) using moments:

\[ c_i = \frac{\langle x_i \theta \rangle}{\langle \theta \rangle} \quad (\partial_t \langle \theta \rangle = 0) \]

\[ \langle x_i \theta \rangle \]

\[ \partial_t \langle x_i \theta \rangle + \langle x_i \nabla \cdot ((\mathbf{U} + \mathbf{x} \cdot \mathbf{A}) \theta) \rangle = \kappa \langle x_i \nabla^2 \theta \rangle \]

\[ \partial_t \langle x_i \theta \rangle - \langle \sum_j A_{ij} \delta_{ij} \theta \cdot \partial x_j \rangle = \kappa \langle \theta \rangle \]

Next moments:

\[ m_{ij} = \frac{\langle x_i x_j \theta \rangle - c_i c_j \langle \theta \rangle}{\langle \theta \rangle} \]

Motion of center of mass

Again, multiply (AD) by \( x_i x_j \) and \( \langle \cdot \rangle \).
\[ \langle x_i x_j \nabla \cdot (u \theta) \rangle = \langle x_i x_j \partial_t \left( (U_h + x_e A h) \theta \right) \rangle \]
\[ = - \langle (U_h + x_e A h) (\delta_i x_j + x_i \delta_j x) \theta \rangle \]
\[ = - U_i c_j \langle \theta \rangle - U_j c_i \langle \theta \rangle - A_{ij} \langle x_i x_j \theta \rangle - A_{ij} \langle x_i x_j \theta \rangle \]
\[ \partial_t (x_i c_j) = c_x \partial_t c_j + c_j \partial_t c_x \]
\[ = c_x (U_j + A_{ij} c_x) + c_j (U_i + A_{ij} c_x) \]
\[ \langle x_i x_j \nabla \cdot (u \theta) \rangle = - \left( \partial_t (x_i c_j) + A_{ij} m_{ij} + A_{ij} m_{ij} \right) \langle \theta \rangle \]
That's the hard part! Next:
\[ \langle x_i x_j \nabla^2 \theta \rangle = \langle \theta \nabla^2 (x_i x_j) \rangle = 2 \langle \theta \rangle \delta_{ij} \]
So finally:
\[ \partial_t m_{ij} = A_{ij} m_{ij} + A_{ij} m_{ij} + 2 \kappa \delta_{ij} \]
Let \((M)_{ij} = m_{ij}\) (symmetric matrix)
\[ \partial_t M = M \cdot A + A^T \cdot M + 2 \kappa I \]

**Moment of inertia equation, spread I patch**

Time to solve these equations!
\[ \xi(t) = \xi(0) \cdot e^{At} + U \cdot \int_0^t e^{A(t-\tau)} d\tau \]
\[ M(t) = e^{A^T t} \cdot M(0) \cdot e^{A t} + 2\kappa \int_0^t e^{A^T(t-\tau)} A(t-\tau) \cdot e^{A \tau} d\tau \]

---

Let \( M = R D R^T \), \( R \) orthogonal, \( D \) diagonal

\[ \dot{M} = \dot{R} D R^T + R \dot{D} R^T + R D \dot{R}^T = R D R^T A + A^T R D R^T + 2\kappa I \]

\[ R^T \dot{R} D + D R^T \dot{R} + \dot{D} = D R^T A R + R^T A^T D R^T + 2\kappa I \]

Now:

\[ \frac{d}{dt} (R^T R) = \dot{R}^T R + R \dot{R} = 0 \]

\[ (R^T R)^T = R^T \dot{R} = -R^T R \]

\( \Rightarrow \) \( R^T R \) is antisymmetric

\[ [R^T R D]_{\hat{\alpha} \hat{\beta}} = (R^T R)_{\hat{\alpha} \hat{\beta}} D_{\hat{\alpha} \hat{\beta}} = (R^T R)_{\hat{\beta} \hat{\alpha}} D_{\hat{\beta} \hat{\alpha}} = 0 \]

(no sum)

\[ \dot{D}_{\hat{\alpha} \hat{\beta}} = D_{\hat{\alpha} \hat{\beta}} \dot{A}_{\hat{\alpha} \hat{\beta}} + \dot{A}_{\hat{\alpha} \hat{\beta}} D_{\hat{\alpha} \hat{\beta}} + 2\kappa \]

\[ \dot{D}_{\hat{\alpha} \hat{\beta}} = 2 \dot{A}_{\hat{\alpha} \hat{\beta}} D_{\hat{\alpha} \hat{\beta}} + 2\kappa \]

Write \( D_{\hat{\alpha} \hat{\beta}} = e^{2p_{\hat{\alpha}} \hat{\beta}} \), with \( p_1 > p_2 > \ldots > p_d \).

\[ \dot{D}_{\hat{\alpha} \hat{\beta}} = 2 e^{2p_{\hat{\alpha}} \hat{\beta}} \dot{p}_{\hat{\alpha}} \hat{\beta} \]

\[ \dot{p}_{\hat{\alpha}} = \dot{A}_{\hat{\alpha} \hat{\beta}} + \kappa e^{-2p_{\hat{\alpha}}} \]
Great equation: $\mathbf{\tilde{A}} = \mathbf{R}^{T} \mathbf{AR} \rightarrow$ rotated velocity gradient matrix.

\[ e^{-2p_i} \rightarrow \text{negligible unless } p_i < 0 \]

Moral: the directions of contraction or compression play an important role.

Now we need an equation for $\mathbf{R}$: off-diagonal terms.

\[
\begin{align*}
[\mathbf{R}^{T} \mathbf{R} \mathbf{D}]_{ij} &= (\mathbf{R}^{T} \mathbf{R})_{ij} \mathbf{D}_{jj} = (\mathbf{R}^{T} \mathbf{R})_{ij} \mathbf{D}_{jj}, \quad i \neq j \\
[\mathbf{D} \mathbf{R}^{T} \mathbf{R}]_{ij} &= \mathbf{D}_{ii} (\mathbf{R}^{T} \mathbf{R})_{ij} = \mathbf{D}_{ii} (\mathbf{R}^{T} \mathbf{R})_{ij} = -(\mathbf{R}^{T} \mathbf{R})_{ij} \mathbf{D}_{jj}
\end{align*}
\]

\[
(\mathbf{D}_{jj} - \mathbf{D}_{ii}) (\mathbf{R}^{T} \mathbf{R})_{ij} = \mathbf{D}_{jj} \mathbf{\tilde{A}}_{ij} + \mathbf{\tilde{A}}_{ij} \mathbf{D}_{jj}
\]

\[
(\mathbf{R}^{T} \mathbf{R})_{ij} = \mathbf{\mathcal{R}}_{ij} \iff \mathbf{\dot{R}} = \mathbf{R} \mathbf{\mathcal{R}}
\]

\[
\mathbf{\mathcal{R}}_{ij} = \frac{e^{2p_i} \mathbf{\tilde{A}}_{ij} + e^{2p_j} \mathbf{\tilde{A}}_{ji}}{e^{2p_i} - e^{2p_j}}
\]

\[ (= 0 \text{ for } i = j) \]

Almost always true for long time, e.g., in 2D, 3D with $p_i + p_j + p_k = 0$, usually a symmetry can break this, or fails locally.

Assume we have separation between the eigenvalues: $e^{2p_i} \gg e^{2p_j}, \ i < j$

\[
\mathbf{\mathcal{R}}_{ij} \approx \frac{e^{2p_i} \mathbf{\tilde{A}}_{ij} + e^{2p_j} \mathbf{\tilde{A}}_{ji}}{e^{2p_i} - e^{2p_j}} = -\mathbf{\tilde{A}}_{ij}, \ i < j
\]
\[ \Omega_{ij} \sim \begin{cases} -\tilde{A}_{ij}, & i < j \\ \tilde{A}_{ij}, & i \geq j \end{cases} \quad \text{(large t)} \]

Independent of eigenvalues!

Can solve: \[ \dot{p}_i = \tilde{A}_{ii} + \kappa e^{-2p_i} \]

\[ p_i(t) = p_{i0} + A_i(t) + \frac{1}{2} \log \left[ 1 + 2\kappa e^{-2p_{i0}} \int_0^t \exp(-2\tilde{A}(t')) dt' \right] \]

where \[ A_i = \int_0^t \tilde{A}_{ii}(t') dt' \]

When diffusion negligible: \[ p_i(t) = p_{i0} + \int_0^t \tilde{A}_{ii}(t') dt' \]

In fact, solving the equation for \( p_i \), \( \kappa = 0 \), is not a bad way of computing Lyapunov exponents:

\[ \lambda_i = \lim_{t \to \infty} \frac{1}{t} p_i(t) \quad \lambda_1 > \lambda_2 > \cdots > \lambda_d \]

Convergence famous

Oseledets Multiplication ergodic theorem

(Some numerical issues regarding orthogonality of \( R \).)

Now comes the stochastic part: could have formulated things

in terms of an SDE. But we take a shortcut:

\[ p_i(t) = p_{i0} + \sum_t \tilde{A}_{ii} \leftarrow \text{sum of uncorrelated random numbers} \]
What is PDF of \( p_x(t) \)?

Recall: if \( x_i \) are i.i.d. and \( X = \sum_{i=1}^{N} x_i \), \( \bar{x} = \frac{\sum x_i}{N} \), \( \bar{x}^2 = \bar{x}^2 = \sigma^2 \)

What is PDF of \( X \)? CENTRAL LIMIT THEOREM

\[
P(X, N) \sim \frac{1}{\sqrt{2\pi N\sigma^2}} \exp\left(-\frac{(X - N\bar{x})^2}{2N\sigma^2}\right)
\]

Valid for: (i) \( N \gg 1 \); (ii) \( X - N\bar{x} < \sqrt{N}\sigma \)

This second restriction is less commonly stated; it tells us that the CLT is not valid in the tails. The CLT tends to vastly underestimate the probability of rare events, or black swans as is trendy to call them these days. These tails matter for mixing.

More generally,

\[
P(X, N) \sim \exp\left(-NS\left(\frac{X - N\bar{x}}{N}\right)\right) \text{ Large deviation form}
\]

\( S(x) \) is a convex function with \( S(0) = S'(0) = 0 \).
\[ S(x) = S'(0) x + \frac{1}{2} S''(0) x^2 + ... \]
\[ S\left(\frac{X-NM^2}{N}\right) = \frac{1}{2} S''(0) \left(\frac{X-NM^2}{N}\right)^2 + ... \]

\[
\exp\left(-NS\left(\frac{X}{N} - \xi\right)^2\right) \sim \exp\left(-\frac{S''(0)}{2N} \left(\frac{X-NM^2}{N}\right)^2\right)
\]

**Compare to CLT:** \[ S''(0) = \frac{1}{\sigma^2} \]

Can also express in terms of \( X \) mean: \( x = \frac{X}{N} \)

\[ P(x, N) \sim \exp\left(-NS(x - \xi)\right) \]

**Example:** Binomial distribution for \( x_i \), \((-1, 1, \text{ mean } 0\) )

\[ p(x_i) = \frac{1}{2} \delta(x_i + 1) + \frac{1}{2} \delta(x_i - 1) \]

\[ e^{-S(x)} = \int p(\xi) e^{-iK\xi} d\xi \]

Characteristic function

\[ = \frac{1}{2} (e^{iK} + e^{-iK}) = \cos K \]
For the mean $x = \frac{1}{N} \sum x_i$:

$$P(x, N) = \int p(x_1) \ldots p(x_N) \delta(x_1 + \ldots + x_N - x) \, dx_1 \ldots dx_N$$

$$e^{-S(k)} = \int P(x, N) e^{-i k x} \, dx$$

$$= \int p(x_1) \ldots p(x_N) e^{-i \frac{k (x_1 + \ldots + x_N)}{N}} \, dx_1 \ldots dx_N$$

$$= \prod_{i=1}^{N} \int p(x_i) e^{-i \frac{k x_i}{N}} \, dx_i \, ^{\text{Inversion Formula}} \, \left( \int p(\xi) e^{-ix_1 \xi/N} \, d\xi \right)^N$$

$$= \left( e^{-S(k/N)} \right)^N = \cos^N \left( \frac{k}{N} \right)$$

$$P(x, N) = \frac{1}{2\pi} \int e^{-S(k)} e^{ikx} \, dk = \frac{1}{2\pi} \int \cos^N \left( \frac{k}{N} \right) e^{ikx} \, dk$$

$$= \frac{N}{2\pi} \int \cos^N k \, e^{iNkx} \, dk, \quad k = \frac{k}{N}.$$  

$$= \frac{N}{2\pi} \int e^{N \left( \log \cos k + iKx \right)} \, dk$$

For $N$ large, look for saddle (stationary) point:

$$\frac{d}{dk} \left( \log \cos k + iKx \right) = -\tan k + ix = 0 \quad \text{when} \quad k = k_{sp}.$$  

$$\tan k_{sp} = -ix$$
\[ H(k, x) = H(k_{sp}, x) + H'(k_{sp}, x)(k-k_{sp}) + \frac{1}{2} H''(k_{sp}, x)(k-k_{sp})^2 + \ldots \]

With this approximation the inverse transform is a Gaussian integral.

Get finally (skip some steps... see Aalto lecture notes)

\[ P(x, N) = \sqrt{\frac{NS''(0)}{2\pi}} e^{-N S(x)}, \quad \text{with} \]

\[ S(x) = -\frac{1}{2} (x+1) \log \left( \frac{1-x}{x+1} \right) + \log(1-x) \quad -1 \leq x \leq 1 \]

Note \( S'(0) = 0 \), \( S'(x) = -\frac{1}{2} \log \left( \frac{1-x}{x+1} \right) \), so \( S'(0) = 0 \)

\[ S''(x) = \frac{1}{1-x^2}, \quad \text{so} \quad S''(0) = 1 \]

\( S'(x) \) is called the rate function. \( S''(x) \) is called the Cramer function. \( S(x) \) is called the entropy function.

For this case the Gaussian form overestimates the probability in the tails (not typical)
What this has to do with mixing?

For $k = 0$, we argued that if $\xi_i$ is a random var., then $p_i$ are distributed according to large deviation form (for large $t$).

$$P(p_1, p_2, t) \sim \exp \left( -t S \left( \frac{p_1 - \lambda_i t}{t} \right) \right) \theta(p_i) \delta(p_1 + p_2)$$

in 2D ($d=2$), (return to 3D later)

ordering incompressibility

$\lambda_i = \lim_{t \to \infty} \frac{p_i}{t} = Lyapunov exp. \geq 0$ (for chaotic flows)

What happens with diffusion? Recall "filament":

The contracting direction "stabilizes" near the Batchelor-width $\sqrt{\frac{\kappa}{\lambda_i}}$. or "freezes"

$$P(p_1, p_2, t) \sim \exp \left( -t S \left( \frac{p_1 - \lambda_i t}{t} \right) \right) P_{\text{stab}}(p_2)$$

stationary distribution.

If we assume, say, an initial Gaussian "patch" of passive scalar, then the concentration at a point scalar $x$

$$\Theta(x, t) \sim \frac{\text{total concentration}}{\text{volume}} \sim (\det M)^{-1/2}$$

$\Theta(x, t) \sim (\det M)^{-1/2}$

$\sum \delta(p_i)$
Expected value:

\[ \langle \Theta^x \rangle (t) \sim \int e^{-xZ_p} \exp \left(-tS \left( \frac{p_i - \lambda, t}{t} \right) \right) \mathcal{P}_{stb} (p_2) \, dp_1 \, dp_2 \]

Non-exponential function of \( t \) (neglect)

\[ \sim \int e^{-x \rho_1} \exp \left(-tS \left( \frac{p_i - \lambda, t}{t} \right) \right) \, dp_1 \]

\[ \int \text{Do the } p_i \text{ integral} \]

Use \( \frac{\rho_i}{t} = \frac{p_i}{t} \) as variable:

\[ \langle \Theta^x \rangle (t) \sim \int e^{-x \rho_1 t} \exp \left(-tS \left( \rho_i - \lambda, t \right) \right) \, d\rho_1 \]

\[ \rho_1 \rightarrow \rho \]

\[ \lambda \rightarrow \lambda_1 \]

\[ \langle \Theta^x \rangle (t) \sim \int e^{-t(\rho_1 + S(\rho_1 - \lambda_1))} \, d\rho \]

Use expected value, not integral
Let $H(h) = \alpha h + S(h-1)$.

For large time, the integral is dominated by saddle point $h^*$:

$H'(h^*) = 0 = \alpha + S'(h^*-1)$

Because of convexity of $S$, $h^*$ is unique.

We then have $H(h) = H(h^*) + \frac{1}{2} H''(h^*)(h-h^*)^2 + ...$

which we use to evaluate the integral. Find:

$\langle \theta^x \rangle (t) \sim e^{-\tau_\alpha t}$, where $\tau_\alpha = H(h^*)$

Note that we do not have $\langle \theta^x \rangle \sim e^{-\tau_\alpha t}$, which would be the case if $\theta$ decayed the same pointwise everywhere.

Kurtosis $\sim \langle \theta^x \rangle \sim e^{-\tau_\alpha t}$

So how do we expect $\tau_\alpha$ to behave?
We have \( \tau_0 = 0 \), since \( S'(h-1) = 0 \) at \( h = \lambda \), and \( S(0) = 0 \).

\[<\theta^*> = 0 \text{ ok!} \]

Hence, \( \tau_* \) changes sign at \( \alpha = 0 \).

What happens for \( \alpha > \alpha_c \)? No saddle point, since would require \( h > 0 \) (not allowed). Hence, take \( h^* = 0 \) (slowest decay).

\[ \tau_* = \alpha \lambda \]

\[ \square \text{ \( \alpha \) constant for } \alpha > \alpha_c \]

\[ b = H(0) = S(-\lambda) \]

Negative powers grow

The moments \( <\theta^*> \) decay slower than expected, all the more so for larger \( \alpha \): \underline{INTERMITTENCY}

Why the leveling-off? For large \( \alpha \), \( <\theta^*> \) is dominated by realizations with large \( \theta \), that is, having experienced little stretching. For \( \alpha > \alpha_c \), these are all that matter, so \( \tau_* \) is the ratio of decay of realizations with no stretching.

\[ p(h,t) \]

\[ p(0,t) \]

\[ \sim e^{-tS(-\lambda)} \]

All this uses for realizations of just one blob, but can scale up to many blobs. (See papers quoted) Validity of theory still controversial, but should work for times that are not too long, scales not too large.
Lecture 4: Mixing in the presence of sources and sinks

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Optimal control vs steepest descent

(From Lin, Thiffeault, Doering.)
Steepest descent of $\dot{H}^{-1}$

(from Lin, Thiffeault, Doering.)
Optimal control vs steepest descent: any flow

(\text{from Lin, Thiffeault, Doering.})
Steepest descent of $\dot{H}^{-1}$: any flow

(from Lin, Thiffeault, Doering.)
Sources and sinks: CO in the atmosphere

Red corresponds to high levels of CO (450 parts per billion) and blue to low levels (50 ppb). Note the immense clouds due to grassland and forest fires in Africa and South America. (Photo NASA/NCAR/CSA.)
function [psi,Effq] = velopt(psi0,src,kappa,q,L,scalefac)

% Problem parameters for Matlab's optimizer fmincon.
psi0 = psi0(:); problem.x0 = psi0(2:end);
problem.objective = @(x) normHq2(x,src,kappa,q,L,scalefac);
problem.nonlcon = @(x) nonlcon(x,src,kappa,q,L,scalefac);
problem.solver = 'fmincon';
problem.options = optimset('Display','iter','TolFun',1e-10,...
    'GradObj','on','GradConstr','on',...
    'algorithm','interior-point');

[psi,Hq2] = fmincon(problem);

% Mixing efficiency: call normHq2 with no flow to get pure-conduction solution.
Effq = sqrt(normHq2(zeros(size(psi)),src,kappa,q,L,scalefac) / Hq2);

psi = reshape([0;psi],size(src)); % Convert psi back into a square grid
function [varargout] = normHq2(psi, src, kappa, q, L, scalefac)

N = size(src,1); src = src(:);

% 2D Differentiation matrices and negative-Laplacian
[Dx, Dy, Dxx, Dyy] = Diffmat2(N, L); mlap = -(Dxx+Dyy);
if q ~= 0 && q ~= -1, error('This code only supports q = 0 or -1.'); end

psi = [0; psi]; ux = Dy*psi; uy = -Dx*psi;
ugradop = diag(sparse(ux))*Dx + diag(sparse(uy))*Dy;

if q == 0
    Aop2 = (-ugradop + kappa*mlap);
elseif q == -1
    Aop2 = mlap*(-ugradop + kappa*mlap);
end
Aop1 = (ugradop + kappa*mlap)*Aop2;

% Solve for chi, dropping corner point to fix normalisation.
chi = [0; Aop1(2:end,2:end) \ src(2:end)];
theta = Aop2*chi;

% The squared $H^q$ norm of theta.
varargout{1} = L^2*sum(theta.^2)/N^2 * scalefac;

if nargout > 1
    % Gradient of squared-norm $H^q$.
    gradHq2 = 2*((Dx*theta).*(Dy*chi) - (Dy*theta).*(Dx*chi));
    varargout{2} = gradHq2(2:end) / N^2 * scalefac;
end
function [c,ceq,gc,gceq] = nonlcon(ksi,src,kappa,q,L,scalefac)

ksi = [0;ksi]; N = size(src,1);
c = []; gc = [];

[Dx,Dy,Dxx,Dyy] = Diffmat2(N,L); % 2D Differentiation matrices
U2 = L^-2*(sum((Dx*ksi).^2 + (Dy*ksi).^2)/N^2);

if nargout > 2
    % Gradient of constraints
    mlappsi = -(Dxx+Dyy)*ksi;
    gceq(:,1) = 2*mlappsi(2:end) / N^2 * scalefac;
end
Left: Optimal stirring velocity field (streamlines) for the source \( \sin x \sin y \), for \( \text{Pe} = 10 \). Right: Dependence on Péclet number of the optimal mixing efficiency \( \varepsilon_0 \). For small \( \text{Pe} \) the optimal streamfunction \( \rightarrow (\sqrt{2\pi})^{-1} \cos x \cos y \).


Lecture 4: Mixing in the Presence of Sources and Sinks

Part 1: Norms

\[ \partial_t \theta + u \cdot \nabla \theta = \kappa \nabla^2 \theta, \quad \nabla \cdot u = 0 \quad \text{(AD)} \]

in \( \Omega \) a \{ \text{bounded domain with zero-flux conditions}, \quad \text{periodic domain} \}

Assume \( \int_{\Omega} \theta \, d\Omega = 0 \). Let \( \| \theta \|_2^2 = \int_{\Omega} \theta^2 \, d\Omega \) \( L^2 \)-norm variance

Recall: \( \frac{d}{dt} \| \theta \|_2^2 = -2 \kappa \| \nabla \theta \|_2^2 \) Equation of variance decay.

Variance \( (L^2\text{-norm}) \) would seem a good measure of mixing. But it requires knowledge of small scales in \( \theta \), which we may not care about. Wouldn’t it be better to blindly solve:

(A) \( \frac{\partial \theta}{\partial t} + u \cdot \nabla \theta = 0 \)?

Since \( \| \theta \|_2 \) we don’t care how something is homogenized.

But then: \( \frac{d}{dt} \| \theta \|_2^2 = 0 \), so can’t use variance.

The advection equation (A) takes us closer to the ergodic theory sense of mixing.
In ergodic theory, we think of an operator $S^t: \Omega \to \Omega$, which is obtained from the solution $\Theta_0(x)$ of (A). $S^t$ "moves forward" a patch of dye $\Theta_0(x)$ to $\Theta(x, t)$.

For a patch $A$, Vol$(A)$ (or Area$(A)$) is the Lebesgue measure of $A$ at $\Theta(x, t)$ only $0$ or $1$, say.

Because of incompressibility, $S^t A$ has the same volume as $A$. $S^t$ is measure-preserving.

Now for the definition of mixing (in the sense of ergodic theory):

$$\lim_{t \to \infty} \text{Vol}(A \cap S^t(B)) = \text{Vol}(A) \text{Vol}(B)$$

for all patches $A, B$ in $\Omega$.

This is called strong mixing. It implies ergodicity, but not the other way around.
Notice that this follows our intuition for what "good mixing" is, but no diffusion is needed.

In fact, the arbitrary "reference patch" $A$ is a bit like a function that we project on. This suggests another def'n, which is more "analytic":

**Weak convergence:**
\[
\lim_{t \to \infty} \langle \Theta(x, t), g \rangle = 0 \quad (\Theta \text{ converges to zero weakly})
\]

for all functions $g(x)$ in $L^2(A)$

Here: $\langle f, g \rangle = \int_A f(x) g(x) \, dx$, and a function $f$ in $L^2(A)$ if $\int_A |f|^2 \, dx < \infty$ (for example, $\delta$-functions are not).

Weak convergence is equivalent to mixing. Why?

\[\Theta(x, t)\]
\[\text{\(\Theta\) is no vanishing, but it is getting wigglier, so} \]
\[\int \Theta \, dx \to 0 \quad (\text{Riemann-Lebesgue lemma})\]

But neither the def'n of mixing and weak convergence are that useful in practice: hard to compute something over all functions $g(x)$!
But there is a simpler way: Mathew, Mezic, & Petzold introduced the mix-norm, which is basically a negative Sobolev norm:

\[ \| \theta \|_{H^q}^2 := \| \nabla^q \theta \|_2^2 \quad \text{Sobolev norm for } H^q(\Omega) \]

\[ (q < 0 \text{ is } L^2 \text{ norm}) \]

We can interpret this norm for \( q < 0 \) as well. This is easiest on a periodic domain:

\[ \| \theta \|_{H^q}^2 = \sum_k \left| \hat{\theta}_k \right|^2 \quad \text{Note } \hat{\theta}_0 = 0 \quad \text{(mean)} \]

For \( q < 0 \), \( \| \theta \|_{H^q} \) smoothes \( \theta \) before taking the \( L^2 \) norm.

**Theorem** (Mathew-Mezic-Petzold, Doering-Lin-T)

\[ \lim_{t \to \infty} \| \theta \|_{H^q} = 0, \; q < 0 \iff \theta \text{ converges weakly to } 0. \]

(proof is short, but a bit technical.)

Upshot: we can track any of these norms to determine if a system is mixing, \( q \) controls how much smoothing is imposed.

This makes optimization easier, for instance.

Time-evolution of \( H^q \)-norms: (w/o diffusion)

\[ \frac{d}{dt} \| \theta \|_{H^q}^2 = \langle \nabla \theta \cdot \nabla u \cdot \nabla \theta \rangle \]

\[ \nabla \sim -i \frac{1}{\beta} \]

\[ \text{NOT conserved even in the absence of diffusion} \]

(Other norms are ugly.)
Mahto, Mezić, Grivopoulos, Vaidya, Petzold: use optimal control to optimize decay of $||\Theta||_H^{2} e^{-\frac{t}{\epsilon}}$ (nonlocal in time)

Lin, Doering, T.: maximize instantaneous decay rate of $||\Theta||_H^{2}$

(local in time, easier, almost as good)

SLIDES pages 2-5
PART 2: SOURCES AND SINKS

\[ \frac{\partial \theta}{\partial t} + u \cdot \nabla \theta = \kappa \nabla^2 \theta + s(x, t) \]  
\( (\nabla \cdot u = 0) \)  
\( (\text{sources/sinks}) \)  
\( > 0 \)  
\( < 0 \)  
\( (\text{SLIDE p.6}) \)

Assume: \( \int s(x, t) \, d\mathcal{V} = 0 \) (otherwise subtract the mean)

More convenient to think of hot/cold sources sinks

For simplicity, restrict to time-independent \( s(x) \).

The system achieves a steady-state, (unlike decaying problem)

\[ u \cdot \nabla \theta = \kappa \nabla^2 \theta + s \]

Let \( \mathcal{L} = u \cdot \nabla - \kappa \nabla^2 \)

\( \mathcal{L} \theta = s \) or \( s = \mathcal{L}^{-1} \theta \)

Integral operator

Note that \( \kappa \neq 0 \) is needed to reach steady-state.

So, assuming the system has reached a steady-state, how do we measure the "quality of mixing"?

Can look at norms \( \| \theta \|_{H^q} \)  
\( (q = 0 \text{ is standard deviation}) \)

But what do we compare to?

One possibility: \[ \frac{\| \theta \|_{H^r}}{\| s \|_{H^s}} \]  
Pretty good, but has units of inverse time.
Prefer mixing enhancement factors:

\[
\mathcal{E}_g = \frac{\|\tilde{\theta}\|_{H^2}}{\|\tilde{\theta}\|_{H^1}}
\]

\[\mathcal{L} = -\kappa \nabla^2\]

\[\mathcal{L} \tilde{\theta} = s\]

\(\tilde{\theta}\) is the solution in the absence of stirring. (purely diffusive)

Since \(\|\tilde{\theta}\|_{H^1}\) is usually decreased by stirring, \(\mathcal{E}_g\) measures the enhancement over the pure-diffusion state.

Several properties given in Doering \& T. Shaw, Doering, \& T.

For instance, can we have \(\mathcal{E}_g < 1\), i.e., can stirring ever be worse than not stirring?

Consider \(\mathcal{E}_1 = \frac{\|\nabla \tilde{\theta}\|_2}{\|\nabla \tilde{\theta}\|_2}\).

\[\tilde{\theta} = \mathcal{L}^{-1} s = (-\kappa \nabla^2)^{-1} s = -\kappa^{-1} \nabla^{-2} s \Rightarrow \nabla \tilde{\theta} = -\kappa^{-1} \nabla^{-1} s\]

Also: \(\mathcal{L} \tilde{\theta} = s \Rightarrow \langle \theta \nabla \tilde{\theta} \rangle = \langle \nabla \tilde{\theta} \rangle \langle \cdot \rangle = \int \nabla \tilde{\theta} \, dx\)

\[\frac{\langle \theta \cdot \nabla \tilde{\theta} \rangle}{\kappa} = \frac{\langle \nabla \tilde{\theta} \cdot \nabla \tilde{\theta} \rangle}{\kappa} = \frac{\langle \nabla \tilde{\theta} \cdot \nabla \tilde{\theta} \rangle}{\kappa} = \frac{\langle \nabla \tilde{\theta} \cdot \nabla \tilde{\theta} \rangle}{\kappa} = \kappa \langle \nabla \tilde{\theta} \cdot \nabla \tilde{\theta} \rangle \]

Also: \(\mathcal{E}_1 = \frac{\|\nabla \tilde{\theta}\|_2}{\|\nabla \tilde{\theta}\|_2}\)
\[ \| \theta \|_{H^1} \leq \| \tilde{\theta} \|_{H^1} \iff \varepsilon_1 \geq 1 \]

This is somewhat counter-intuitive: gradients are usually increased by stirring! However, here we're talking about gradients in a steady-state, affected by diffusion.

What about the other ones, \( \varepsilon_q, q \neq 1 \)? Do we have \( \varepsilon_q \geq 1 \)?

We tried and failed to prove this because it isn't true. Following a challenge by Charlie Doering, Jeff Weiss came up with something like:

\[
\begin{align*}
\eta &= (2 \sin x \cos 2y, -\cos x \sin 2y) \\
\nu &= (\cos x - \frac{1}{2}) \sin y
\end{align*}
\]

(Péclet = 4)

This manages to "concentrate" the source sink distribution more than under pure diffusion, and

\[
E_0 \approx 0.978, \quad \varepsilon_1 \approx 0.945
\]

Slightly less than 1! Not a dramatic effect, but it's there!

(more later)
Optimization: What kinds of flow give the largest Eq, given source/sink distribution $s(x)$? (Fixed energy)

Surprising example: $s(x) = \sin x$ (periodic B.C.)

Optimal: $u = U\hat{x}$ \text{ Constant flow!}

(see Shaw-T-Doering, Plasting-Yang)

This example demonstrates that with body sources the best stirring has more to do with transport than with creation of small scales.

Solve numerically for more complicated sources. (Slide p.7, Matlab)