Lecture 5: Stochastic Models (part 2)

(moment of inertia tensor)

\[ M = M \cdot A + A^T \cdot M + 2\kappa I \]

where \( M = RDR^T \) is orthogonal, \( D \) diagonal.

Eigenvalues \( \rho_1 \geq \rho_2 \geq \ldots \geq \rho_d \).

\[ \dot{\rho}_i = \tilde{A}_{ii} + \kappa e^{-2\rho_i} \]

\[ \tilde{A}_{ii} = R^T A R, \] evolves independent for \( \rho_1 \gg \rho_2 \gg \ldots \gg \rho_d \).

\[ \rho_i(t) = \rho_{i0} + A_i(t) + \frac{1}{2} \log \left[ 1 + 2\kappa e^{-2\rho_i} \int_0^t \exp(-2A_i(t')) dt' \right] \]

where

\[ A_i = \int_0^t \tilde{A}_{ii}(t') dt' \]

(see Falkovich et al. for description as SDE)

For \( \kappa = 0 \), we argued that if \( \tilde{A}_{ii} \) is a random var., then \( \rho_i \) are distributed according to large deviation form (for large \( t \)):

\[ P(\rho_1, \rho_2, t) \sim \exp \left( -t S(\rho_i - A_i(t)) \right) \theta(\rho_i) \delta(\rho_1 + \rho_2) \]

in 2D (\( d = 2 \), return 3D later)

\( \lambda_1 = \lim_{t \to \infty} \frac{\rho_1}{t} = \text{Lyapunov exp.} \]

\( \theta(x) = \begin{cases} 1 & \text{step function} \\ 0 & \text{for chaotic flow} \end{cases} \)

\( \rho_1 \geq \rho_2 \)
What happens with diffusion? Recall "filament":

The contracting direction "stabilizes" near the Batchelor width \( \sqrt{\frac{\nu}{\lambda_1}} \) or "freezes".

\[
\begin{align*}
\mathbb{P}(p_1, p_2, t) &\sim \exp \left( -t S \left( \frac{p_i - \lambda_1 t}{t} \right) \right) \mathbb{P}_{\text{stb}} (p_2) \\
\Theta(x, t) &\sim \frac{\text{total concentration}}{\text{volume}} \sim (\det M)^{-1/2} \\
&\quad \text{independent of } x
\end{align*}
\]

\( \exp (-\sum_i p_i) \)

Expected value:

\[
\langle \Theta^x \rangle (t) \sim \int_e^{e^{-\sum p_i}} \exp(-S(p_i - \lambda_1, t)) \mathbb{P}_{\text{stb}} (p_2) \, dp_1 \, dp_2
\]

Non-exponential function of \( t \) (neglect) \( \sim \int \exp(-S(p_i - \lambda_1, t)) \, dp_1 \left\rangle \text{Do the } p_i \text{ integral} \right. \\

Use \( h_i = p_i / t \) as variable:

\[
\langle \Theta^x \rangle (t) \sim \int e^{-\sum h_i} e^{-t S(h_i - \lambda_1, t)} \, dh_i
\]

Use \( h_i = \frac{p_i}{t} \) as variable:

\[
\langle \Theta^x \rangle (t) \sim \int e^{-t (\lambda_1 - S(h - \lambda_1))} \, dh
\]

\( \lambda_1 \to \lambda \)

\( h \to h_1 \)
Let \( H(h) = \alpha h + S(h-1) \).

For large time, the integral is dominated by saddle point \( h^* \):
\[ H'(h^*) = 0 = \alpha + S'(h^* - 1) \]

Because of convexity of \( S \), \( h^* \) is unique.

We then have \( H(h) = H(h^*) + \frac{1}{2} H''(h^*) (h-h^*)^2 + \ldots \)

which we use to evaluate the integral. Find:
\[ \langle \theta^x \rangle(t) \sim e^{-\tau_\alpha t} \]
where \( \tau_\alpha = H(h^*) \)

Note that we do not have \( \langle \theta^x \rangle \sim e^{-\alpha \tau_\alpha t} \) which would be the case if \( \theta \) decayed the same pointwise everywhere.

Kurtosis \( \sim \frac{\langle \theta^x \rangle}{\langle \theta \rangle^2} \sim e^{-\tau_\alpha t} \)

So how do we expect \( \tau_\alpha \) to behave?
We have \( \tau_0 = 0 \), since \( S'(h-1) = 0 \) at \( h = \lambda \), and \( S(0) = 0 \).

(\( \langle \theta^0 \rangle = 0 \) ok!)

Hence, \( \tau_2 \) changes sign at \( \alpha = 0 \).

What happens for \( \alpha > \alpha_c \)? No saddle point, since would require \( h < 0 \) (not allowed). Hence, take \( h^* = 0 \) (slowest decay).

\[ \tau_0 = \alpha^2 \quad \text{\( \tau_2 \) constant for \( \alpha \geq \alpha_c \).} \]

\[ b = H(a) = S'(a) \]

The moments \( \langle \theta^x \rangle \) decay slower than expected, all the more so for larger \( \alpha \): **INTERMITTENCY**

Why the leveling off? For large \( \theta \), \( \langle \theta^x \rangle \) is dominated by realizations with large \( \theta \), that is, having experienced little stretching. For \( \alpha > \alpha_c \), these are all that matter, so \( \tau_2 \) is the rate of decay of realizations with no stretching.

\[ p(h,t) \]

\[ p(0,t) \sim e^{-t S'(a)} \]

All this use for realizations of just one blob, but can scale up to many blobs. (See papers quoted) Validity of theory still controversial, but should work for times that are not too long, scales not too large.