Chaotic Mixing and Lagrangian Coordinates

Jean-Luc Thiffeault

Department of Applied Physics and Applied Mathematics

Columbia University

http://plasma.ap.columbia.edu/~jeanluc

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The Advection-diffusion Equation

$$\frac{\partial\phi}{\partial t} + \mathbf{v}\cdot\boldsymbol{\nabla}\phi = \frac{1}{\rho}\,\boldsymbol{\nabla}\cdot(\rho\,\mathbf{D}\,\boldsymbol{\nabla}\phi)$$

- $\mathbf{v}(\mathbf{x},t)$ — Eulerian velocity field
- $\phi(\mathbf{x},t)$ — concentration of passive scalar
- $\mathbf{D}(\mathbf{x},t)$ — diffusivity tensor $(D/vL \ll 1)$
- $\rho(\mathbf{x},t)$ — density

ϕ could be temperature, or the concentration of a reacting chemical, or…

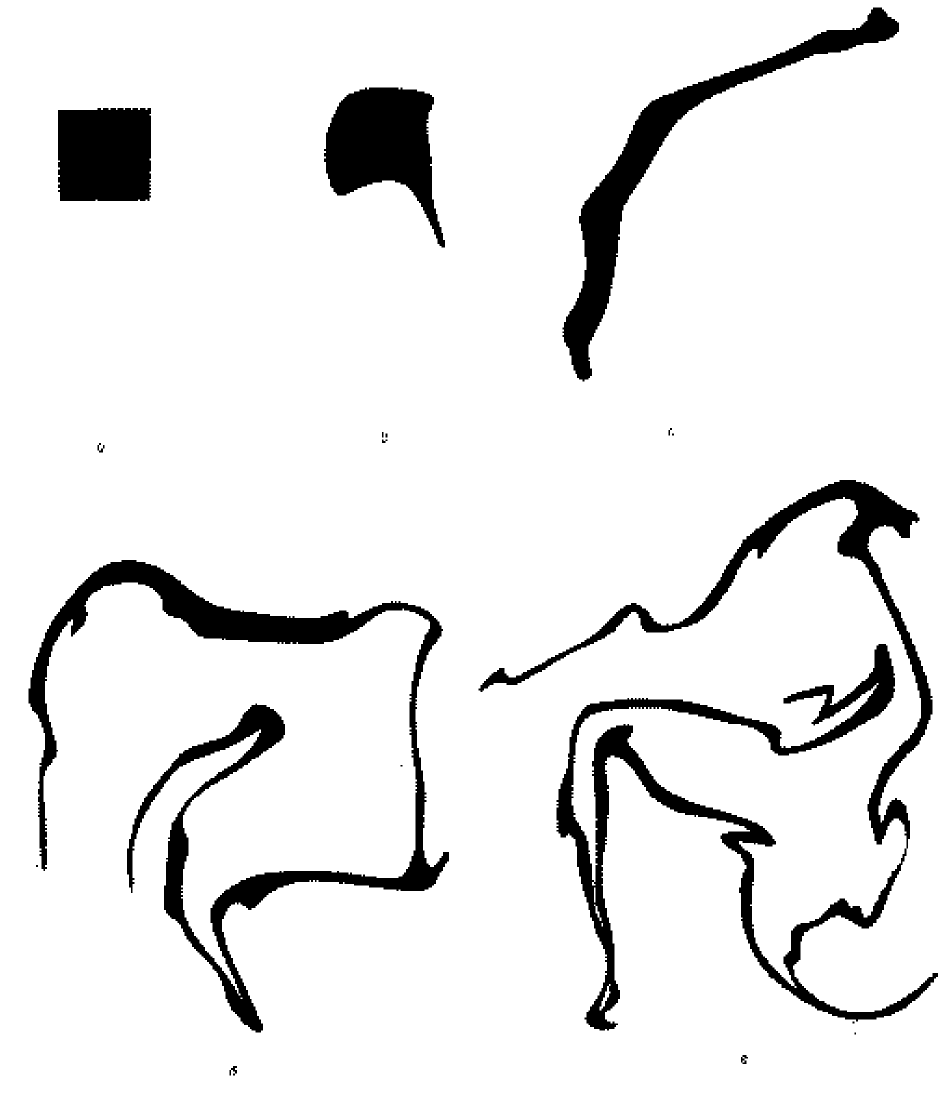
Small diffusivity is the norm rather than the exception.

Typical values of D/vL:

Core of earth	10^{-3}
Temperature in a room	10^{-10}
Solar corona	10^{-12}
Galaxy	10^{-19}

Even a tiny amount of diffusivity matters.

Chaotic Stirring



Chaotic Mixing

- Strain in the velocity field generates small scales, even for nonturbulent flows
- Huge gradients of ϕ are created
- Makes enhanced diffusion possible:
  For heat in a room, turns a diffusion time of months into minutes (exponential)
- Very difficult to simulate directly : scale separation $\sim 10^{10}$
- Lagrangian (comoving) coordinates are very convenient because the chaos gets “hidden” in the coordinate transformation.
- Differential geometry provides a novel perspective.

Overview

- In a fluid flow, Lagrangian coordinates label fluid elements. The Lagrangian frame moves and stretches with the flow.
- When the flow is chaotic, Lagrangian quantities that characterize the geometry and dynamics of the system have a well-defined asymptotic behavior: Lyapunov exponents, characteristic directions…
- Useful even for "short" times: finite-time Lyapunov exponents. Characteristic directions converge very quickly.
- The study of these Lagrangian quantities leads to some surprising results: they obey constraints due to the chaotic nature of the flow, which leads to a one-dimensional diffusion equation in Lagrangian coordinates.

Lagrangian Coordinates

Trajectory of a fluid element in Eulerian coordinates $\mathbf{x}$

$$\frac{d\mathbf{x}}{dt}(\mathbf{a}, t) = \mathbf{v}(\mathbf{x}(\mathbf{a}, t), t)$$

$\mathbf{a}$ are Lagrangian coordinates that label fluid elements.

$\mathbf{x}(\mathbf{a}, t = 0) = \mathbf{a}$: fluid elements are labeled by their initial position.

$\mathbf{x} = \mathbf{x}(\mathbf{a}, t)$ is thus the (smooth) transformation from Lagrangian ($\mathbf{a}$) to Eulerian ($\mathbf{x}$) coordinates.

For a chaotic flow, this transformation gets horrendously complicated as time evolves.

The Metric Tensor

The Jacobian matrix of the transformation $\mathbf{x}(\mathbf{a}, t)$ is

$$M^i{}_q \equiv \frac{\partial x^i}{\partial a^q}$$

Restrict ourselves to incompressible flows, $\boldsymbol{\nabla} \cdot \mathbf{v} = 0$, so that $\det M = 1$.

Jacobian matrix is a precise record of how a fluid element is rotated and stretched by $\mathbf{v}$.

Interested in the stretching, not the rotation, so we construct the metric tensor

$$\boxed{g_{pq} \equiv \sum_{i=1}^{n} M^i{}_p \, M^i{}_q}$$

which contains only the information on the stretching of fluid elements.

Stretching and Contracting Directions

Metric is a symmetric, positive-definite matrix $\Longrightarrow$ can be locally diagonalized with orthogonal eigenvectors $\{\hat{\mathbf{e}}_\sigma\}$ and corresponding real, positive eigenvalues $\{\Lambda_\sigma^2\}$,

$$g_{pq} = \sum_{\sigma=1}^{n} \Lambda_\sigma^2 \, (\hat{\mathbf{e}}_\sigma)_p \, (\hat{\mathbf{e}}_\sigma)_q$$

The Λ_σ are called coefficients of expansion and are ordered such that $\Lambda_1 > \Lambda_2 > \ldots > \Lambda_n$ [assumed nondegenerate].

The Λ_σ are related to the finite-time Lyapunov exponents λ_σ by

$$\lambda_\sigma = \log \Lambda_\sigma / t$$

The incompressibility of $\mathbf{v}$ implies that $\Lambda_1 \, \Lambda_2 \cdots \Lambda_n = 1$.

The label u indicates the most unstable direction:

$$\hat{\mathbf{e}}_1 \equiv \hat{\mathbf{u}} \ , \qquad \Lambda_1 \equiv \Lambda_u$$

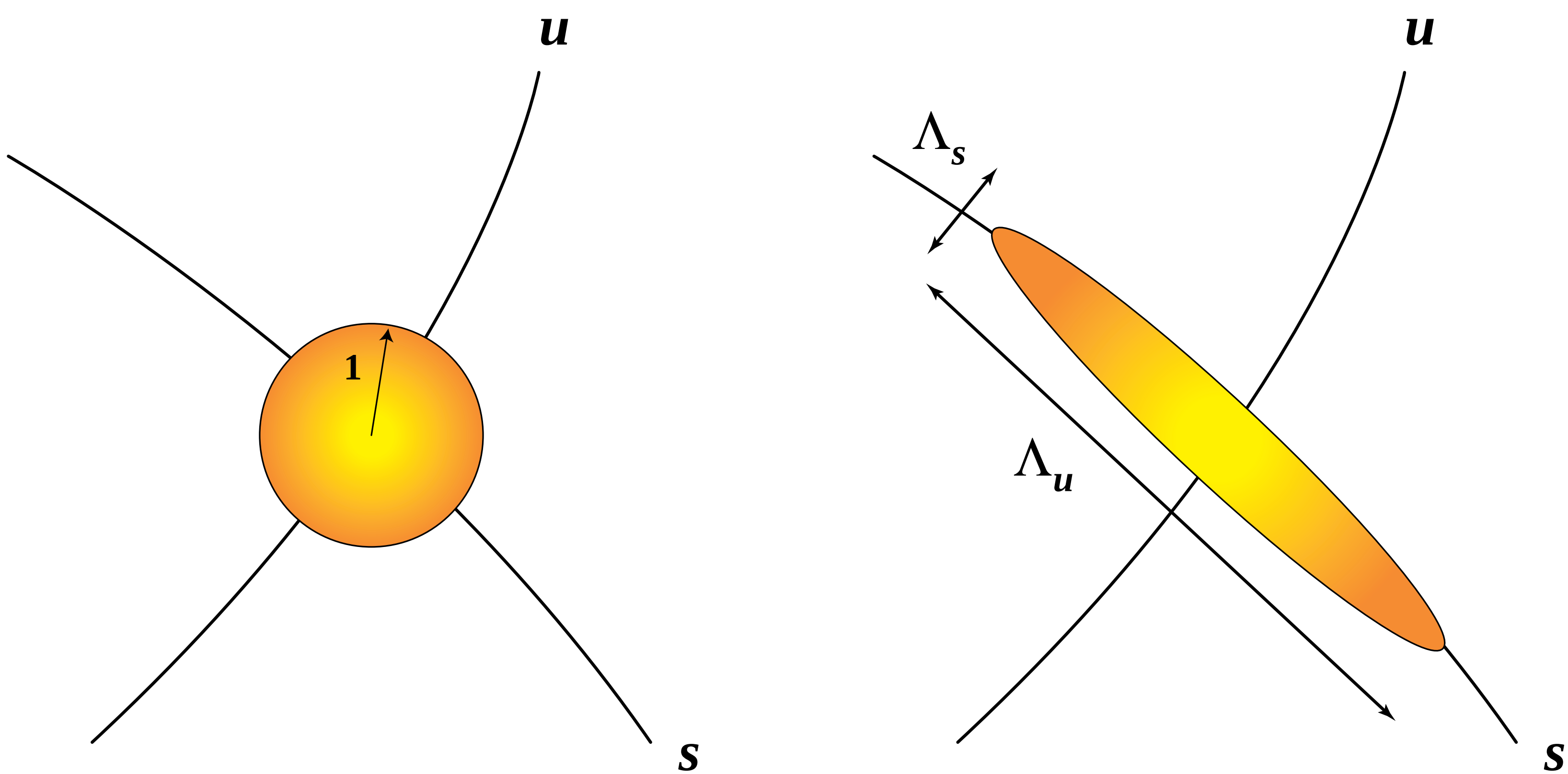
After some time, $\Lambda_u \gg 1$, growing exponentially for long times.

The label s indicates the most stable direction:

$$\hat{\mathbf{e}}_n \equiv \hat{\mathbf{s}} \ , \qquad \Lambda_n \equiv \Lambda_s$$

After some time, $\Lambda_s \ll 1$, decreasing exponentially for long times.

The eigenvalues and eigenvectors describe the deformation of a fluid element in a comoving frame:

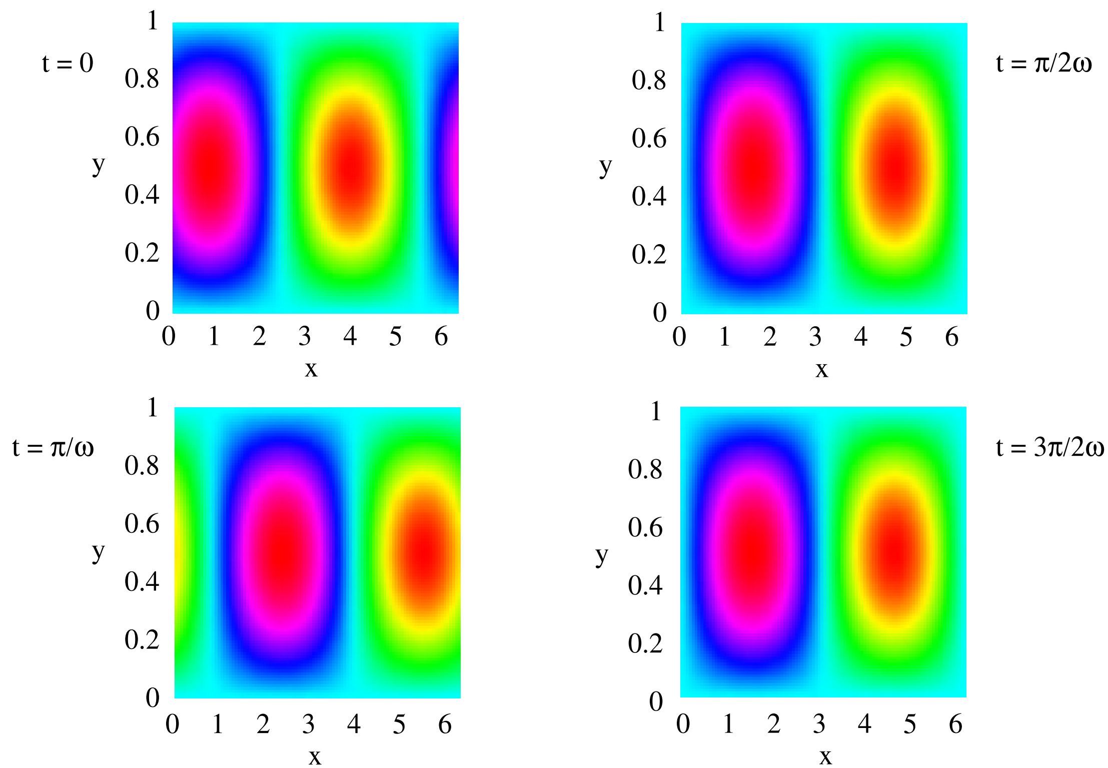


The $\hat{\mathbf{u}}$ and $\hat{\mathbf{s}}$ directions can be integrated to yield the unstable and stable manifolds.

Example: 2D Convection Rolls

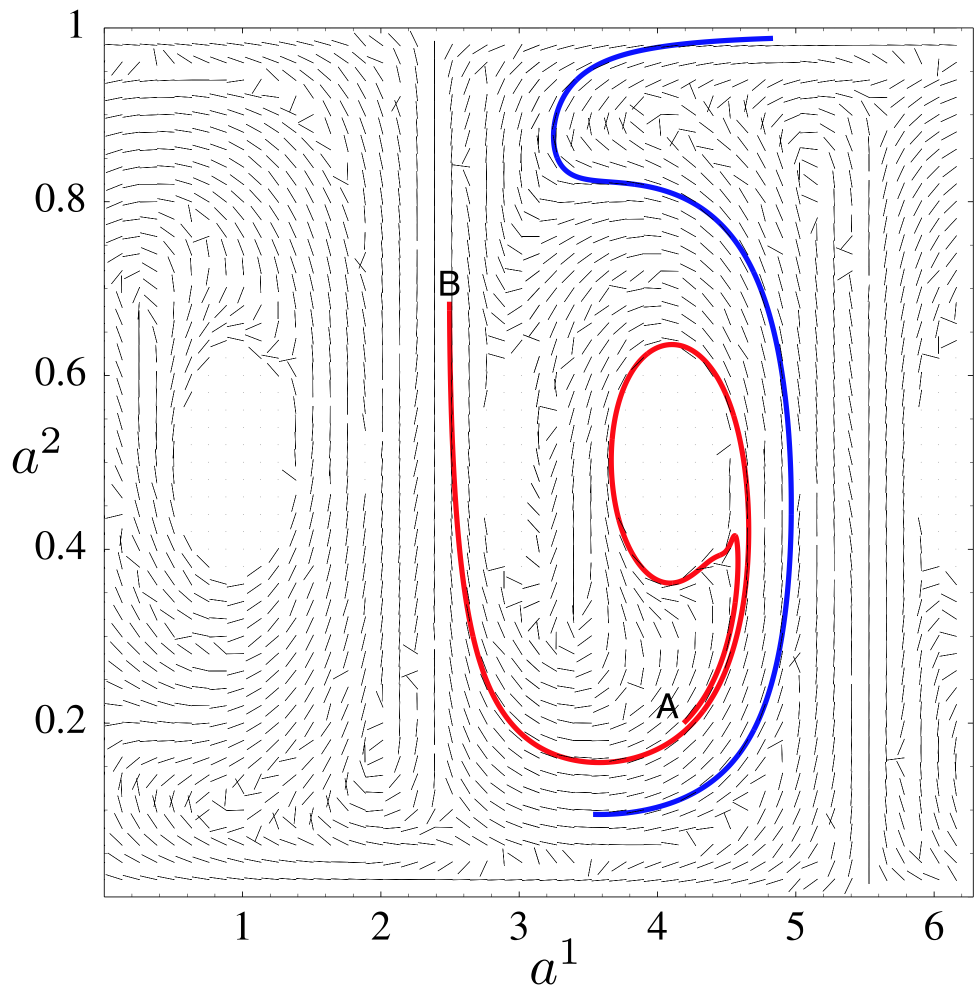
Oscillating convection rolls: $\mathbf{v} = (-\partial_y \psi, \partial_x \psi)$, with

$$\psi(\mathbf{x}, t) = Ak^{-1}(\sin kx \sin \pi y + \epsilon \cos \omega t \cos kx \cos \pi y)$$



[Movie: lyap_crolls_jet.mpeg]

- The magnitude of the Lyapunov exponents is the time-averaged straining rate encountered by a fluid element.
- More red at the beginning because some fluid elements are drawn to the hyperbolic points, where the strain is maximal.
- However, those fluid elements don't stay there long, and there is an equilibration toward blue (lowest stretching).
- Central region with low mixing, corresponding to the center of the rolls. Blue filaments extend outside the rolls.
- Study breakup of oil droplets [Solomon].



$\hat{\mathbf{s}}$ field for oscillating rolls. Two typical portions of stable manifolds in red and blue. Motion in central region is nonchaotic.

ABC Flow

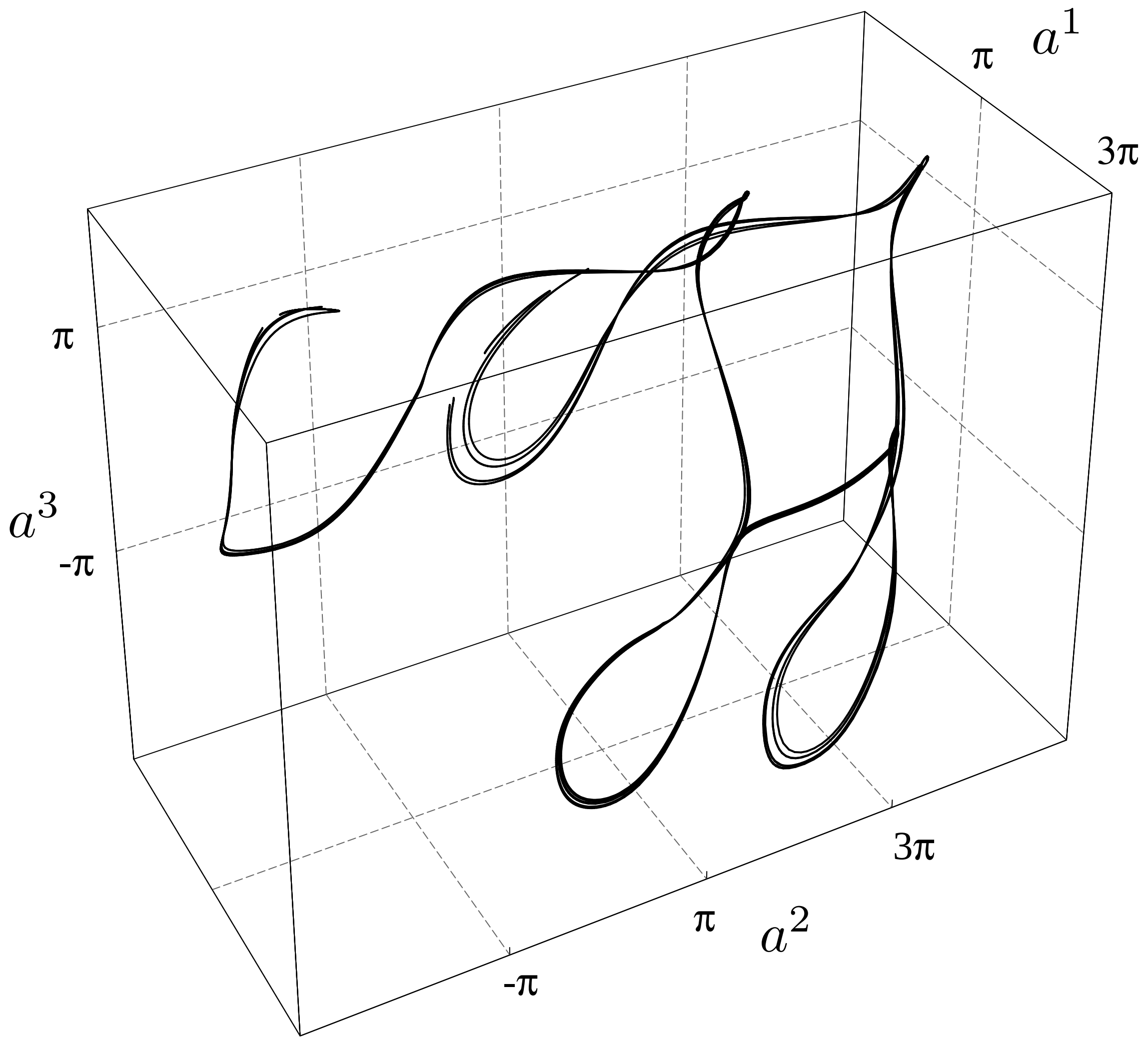
Another well-known model system is the ABC flow,

$$\mathbf{v}(\mathbf{x}) = A\,(0, \sin x_1, \cos x_1) + B\,(\cos x_2, 0, \sin x_2) + C\,(\sin x_3, \cos x_3, 0)$$

a sum of three Beltrami waves, which satisfy $\boldsymbol{\nabla} \times \mathbf{v} \propto \mathbf{v}$. It is time-independent and incompressible ($|g| = 1$).

We shall be using the parameter values $A = 5, B = C = 2$ in subsequent examples.

Here's a portion of a stable manifold $\mathbf{s}(\mathbf{a})$ for the ABC 522 flow:



Advection–Diffusion: Lagrangian Picture

In Lagrangian coordinates, the advection-diffusion equation is

$$\left.\frac{\partial\phi}{\partial t}\right|_{\mathbf{a}} = \sum_{p,q} D\,\frac{\partial}{\partial a^p}\left[g^{pq}\,\frac{\partial\phi}{\partial a^q}\right]$$

where $g^{pq} = (g^{-1})^{pq}$. Assuming $\Lambda_s \ll 1$, can approximate by

$$\left.\frac{\partial\phi}{\partial t}\right|_{\mathbf{a}} = \sum_{p,q} D\,\frac{\partial}{\partial a^p}\left[\Lambda_s^{-2}\,\hat{s}^p\,\hat{s}^q\,\frac{\partial\phi}{\partial a^q}\right]$$

Define: $\tilde{\mathbf{s}} \equiv \Lambda_s^{-1}\,\hat{\mathbf{s}}$,

$$\left.\frac{\partial\phi}{\partial t}\right|_{\mathbf{a}} = \sum_{p,q} D\,\frac{\partial}{\partial a^p}\left[\tilde{s}^p\,\tilde{s}^q\,\frac{\partial\phi}{\partial a^q}\right]$$

Not quite a 1–D diffusion equation…

Constraint, Type I

It was shown that arbitrary chaotic flows satisfy several differential constraints, one of which can be written

$$\sum_p \left(\frac{\partial}{\partial a^p}\, \hat{s}^p - \hat{s}^p \, \frac{\partial}{\partial a^p} \, \log \Lambda_s \right) \longrightarrow 0,$$

where s denotes any contracting direction.

In terms of $\tilde{\mathbf{s}}$, can write the constraint as

$$\sum_p \frac{\partial}{\partial a^p}\, \tilde{s}^p = 0.$$

[Tang & Boozer 1996, Thiffeault & Boozer 2001, Thiffeault 2001]

Using this constraint in our advection-diffusion equation, we find

$$\left.\frac{\partial \phi}{\partial t}\right|_{\mathbf{a}} = \sum_{p,q} \widetilde{D}(t)\, \tilde{s}^p\, \frac{\partial}{\partial a^p} \left[\tilde{s}^q \frac{\partial \phi}{\partial a^q}\right]$$

or

$$\boxed{\left.\frac{\partial \phi}{\partial t}\right|_{\mathbf{a}} = \widetilde{D}(t)\, \frac{\partial^2 \phi}{\partial s^2}} \qquad \text{where} \quad \frac{\partial}{\partial s} \equiv \sum_{p} a^p \frac{\partial}{\partial a^q}$$

This is a bona-fide one-dimensional diffusion equation with a time-dependent diffusion coefficient!

Raises the possibility of solving the advection-diffusion equation in the small diffusivity limit (the difficult one).

[Submitted to Physical Review Letters]

Summary

- In the small diffusivity limit, the advection-diffusion equation cannot be solved directly because of scale separation.
- Lagrangian coordinates are a powerful tool for studying chaotic flows, because inessential information can be discarded.
- These derivatives are not all independent and must obey constraints due to the exponential behavior in chaotic flows. The constraints have consequences in physical problems.
- For example, one type of constraint helps reduce the advection-diffusion equation to 1 dimension.