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# Eulerian and Lagrangian Pictures of Mixing

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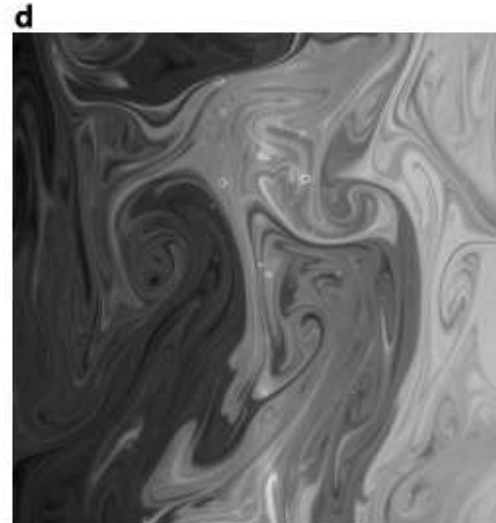
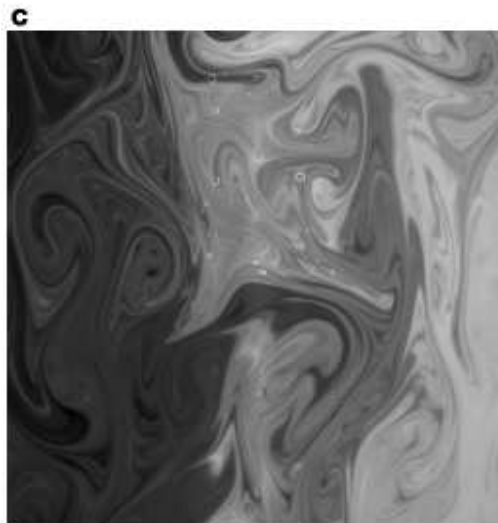
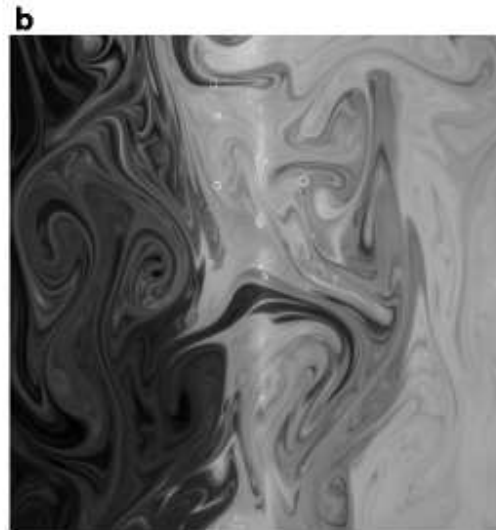
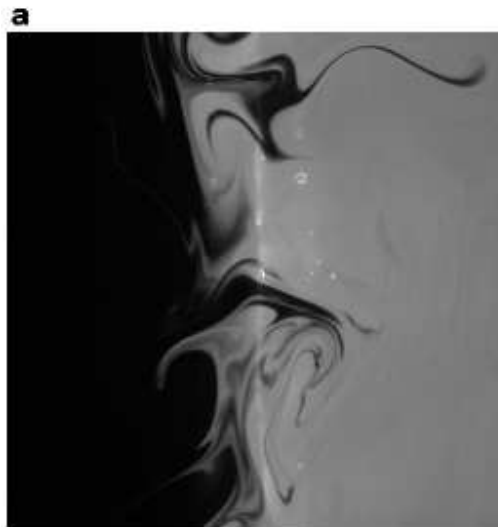
Steve Childress

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# Experiment of Rothstein *et al.*: Persistent Pattern



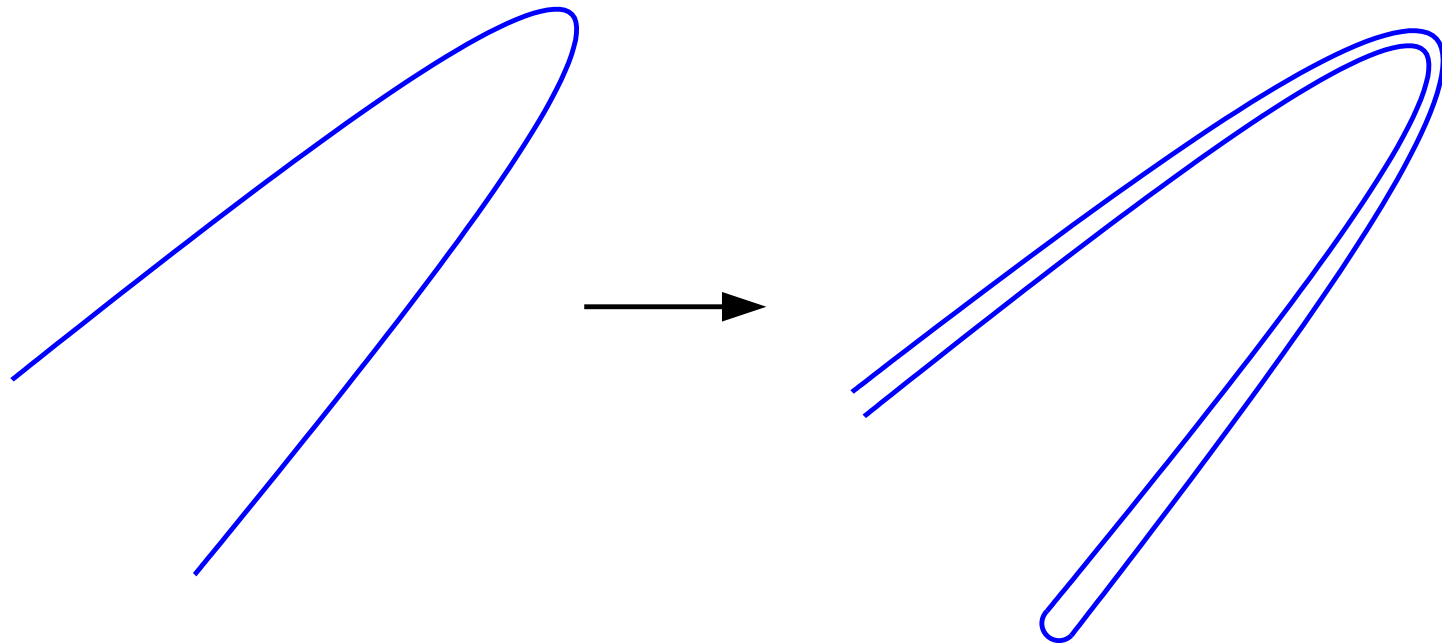
Disordered array of magnets with oscillatory current drive a thin layer of electrolytic solution.

periods 2, 20, 50, 50.5

[Rothstein, Henry, and Gollub,  
Nature **401**, 770 (1999)]

# Evolution of Pattern

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- “Striations”
- Smoothed by diffusion
- Eventually settles into “pattern” (eigenfunction)

# Local vs Global Regimes of Mixing

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Local theory:

- Based on distribution of Lyapunov exponents.

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Average over angles

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## Global theory:

- Eigenfunction of advection–diffusion operator.
- [Pierrehumbert, Chaos Sol. Frac. (1994)]      Strange eigenmode  
[Fereday et al., Wonhas and Vassilicos, PRE (2002)]      Baker's map  
[Sukhatme and Pierrehumbert, PRE (2002)]  
[Fereday and Haynes (2003)]      Unified description

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- **Today**: Try to connect the two pictures.
- Cannot often do this! Map allows (**mostly**) analytical results.

# A Bit of History

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Eulerian (**spatial**) coordinates are due to...

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d'Alembert

# A Bit of History

---

... and Lagrangian (**material**) coordinates to...



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d'Alembert



Euler

---

The people responsible for the confusion. . .

---

The people responsible for the confusion...



Lagrange



Dirichlet

(See footnote in Truesdell, *The Kinematics of Vorticity*.)

# The Map

We consider a diffeomorphism of the 2-torus  $\mathbb{T}^2 = [0, 1]^2$ ,

$$\mathcal{M}(\mathbf{x}) = \mathbb{M} \cdot \mathbf{x} + \phi(\mathbf{x}),$$

where

$$\mathbb{M} = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}; \quad \phi(\mathbf{x}) = \frac{\varepsilon}{2\pi} \begin{pmatrix} \sin 2\pi x_1 \\ \sin 2\pi x_1 \end{pmatrix};$$

$\mathbb{M} \cdot \mathbf{x}$  is the **Arnold cat map**.

The map  $\mathcal{M}$  is **area-preserving** and **chaotic**.

For  $\varepsilon = 0$  the stretching of fluid elements is **homogeneous in space**.

For small  $\varepsilon$  the system is still **uniformly hyperbolic**.

# Advection and Diffusion: Eulerian Viewpoint

Iterate the map and apply the **heat operator** to a scalar field (which we call **temperature** for concreteness) distribution  $\theta^{(i-1)}(\mathbf{x})$ ,

$$\theta^{(i)}(\mathbf{x}) = \mathcal{H}_\kappa \theta^{(i-1)}(\mathcal{M}^{-1}(\mathbf{x}))$$

where  $\kappa$  is the **diffusivity**, with the **heat operator**  $\mathcal{H}_\kappa$  and **kernel**  $h_\kappa$

$$\mathcal{H}_\kappa \theta(\mathbf{x}) := \int_{\mathbb{T}^2} h_\kappa(\mathbf{x} - \mathbf{y}) \theta(\mathbf{y}) \, d\mathbf{y};$$

$$h_\kappa(\mathbf{x}) = \sum_{\mathbf{k}} \exp(2\pi i \mathbf{k} \cdot \mathbf{x} - \mathbf{k}^2 \kappa).$$

In other words: **advect** instantaneously and then **diffuse** for one unit of time.

# Transfer Matrix

Fourier expand  $\theta^{(i)}(\mathbf{x})$ ,

$$\theta^{(i)}(\mathbf{x}) = \sum_{\mathbf{k}} \hat{\theta}_{\mathbf{k}}^{(i)} e^{2\pi i \mathbf{k} \cdot \mathbf{x}}.$$

The effect of advection and diffusion becomes

$$\hat{\theta}_{\mathbf{k}}^{(i)}(\mathbf{x}) = \sum_{\mathbf{q}} \mathbb{T}_{\mathbf{k}\mathbf{q}} \hat{\theta}_{\mathbf{q}}^{(i-1)},$$

with the [transfer matrix](#),

$$\begin{aligned} \mathbb{T}_{\mathbf{k}\mathbf{q}} &:= \int_{\mathbb{T}^2} \exp\left(2\pi i (\mathbf{q} \cdot \mathbf{x} - \mathbf{k} \cdot \mathcal{M}(\mathbf{x})) - \kappa \mathbf{q}^2\right) d\mathbf{x}, \\ &= e^{-\kappa \mathbf{q}^2} \delta_{0, Q_2} i^{Q_1} J_{Q_1}((k_1 + k_2) \varepsilon), \quad \mathbf{Q} := \mathbf{k} \cdot \mathbb{M} - \mathbf{q}, \end{aligned}$$

where the  $J_Q$  are the Bessel functions of the first kind.

# Variance: A Measure of Mixing

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In the absence of diffusion ( $\kappa = 0$ ) the **variance**  $\sigma^{(i)}$

$$\sigma^{(i)} := \int_{\mathbb{T}^2} |\theta^{(i)}(\mathbf{x})|^2 d\mathbf{x} = \sum_{\mathbf{k}} \sigma_{\mathbf{k}}^{(i)}, \quad \sigma_{\mathbf{k}}^{(i)} := |\hat{\theta}_{\mathbf{k}}^{(i)}|^2$$

is **preserved**. (We assume the spatial mean of  $\theta$  is zero.)

For  $\kappa > 0$  the variance **decays**.

We consider the case  $\kappa \ll 1$ , of greatest practical interest.

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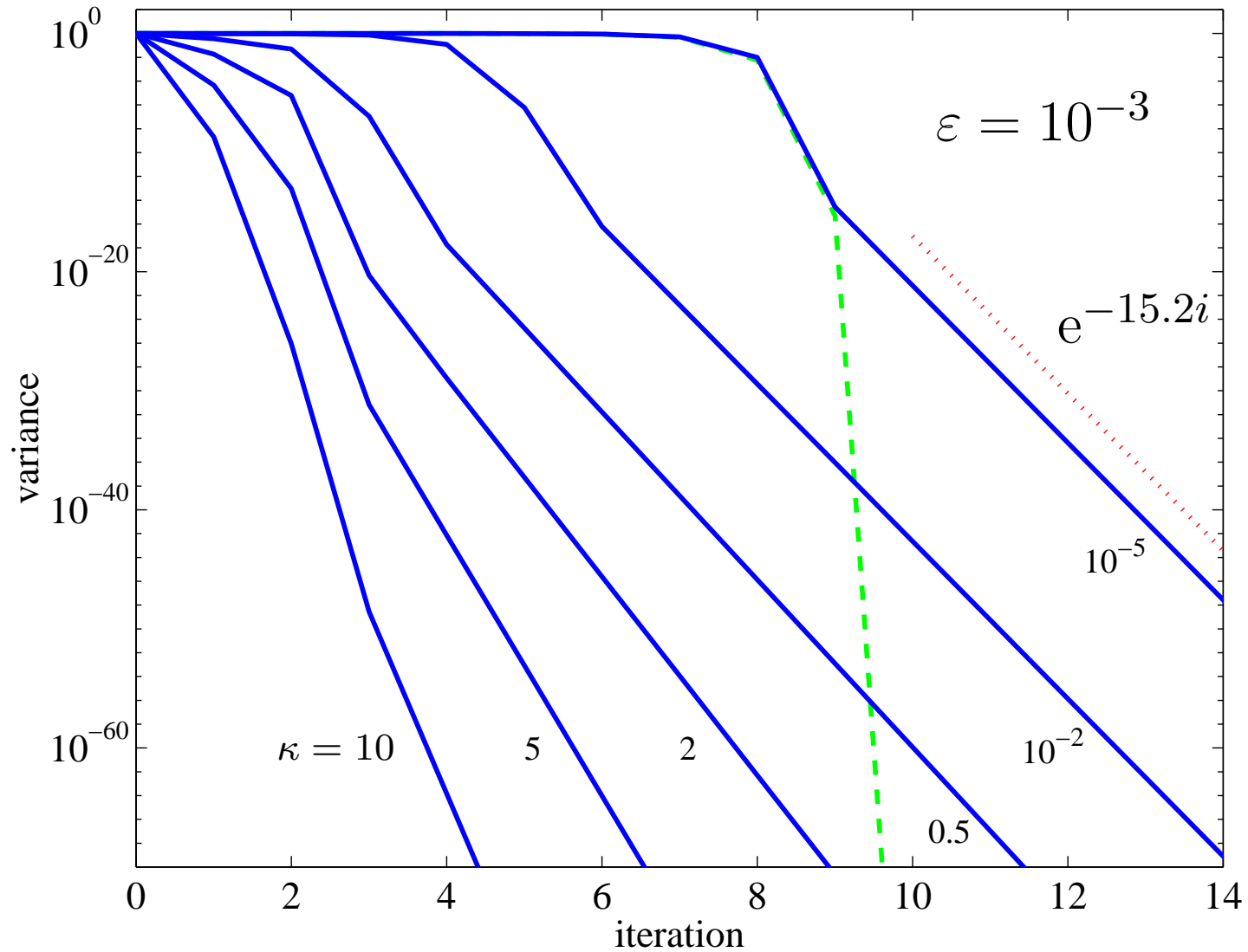
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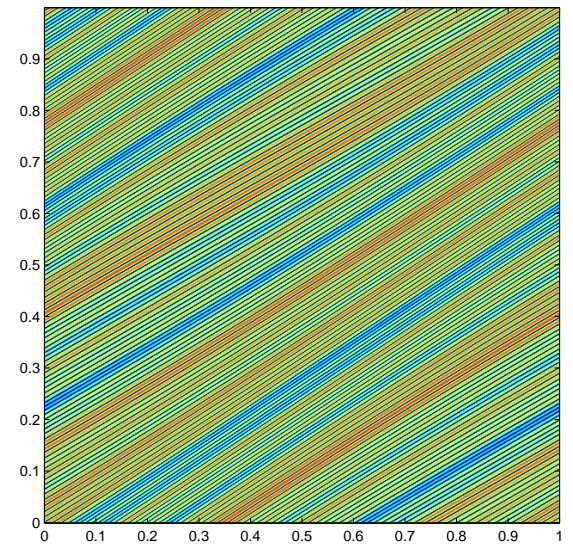
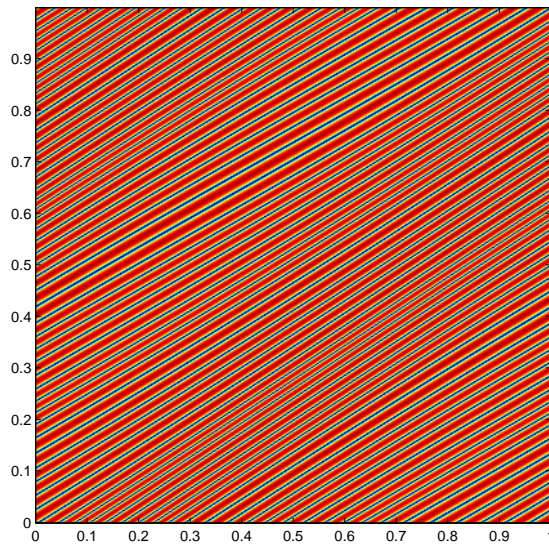
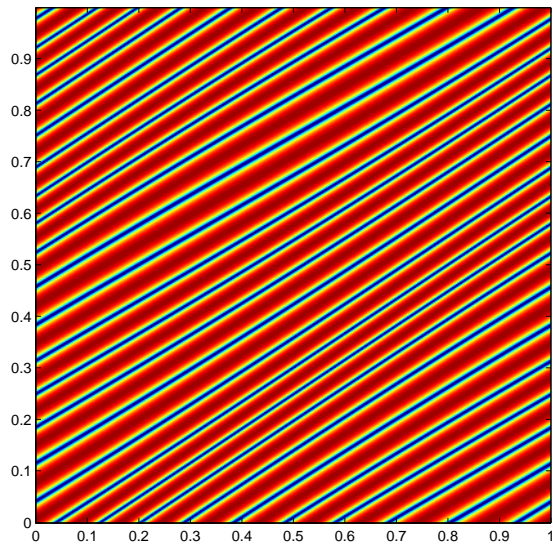
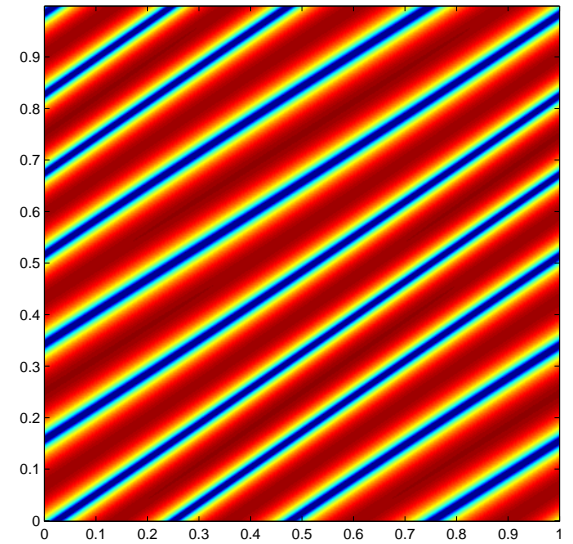
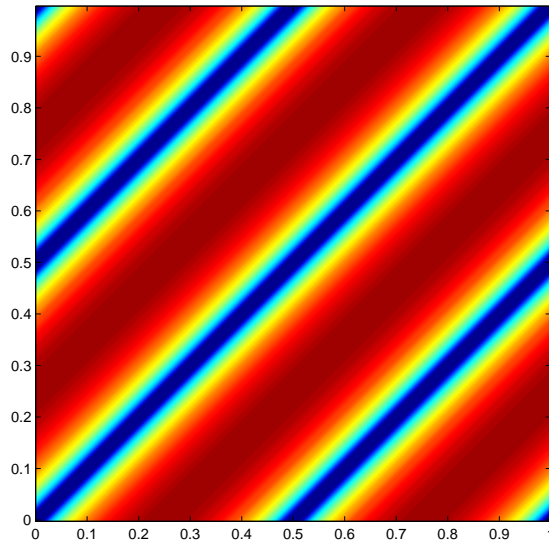
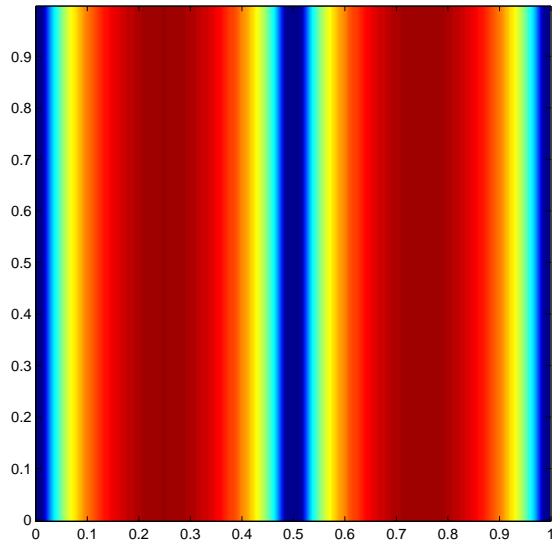
**Three phases:**

- The variance is initially **constant**;
- It then undergoes a rapid **superexponential** decay;
- $\theta^{(i)}$  settles into an eigenfunction of the A–D operator that sets the **exponential** decay rate.

# Decay of Variance

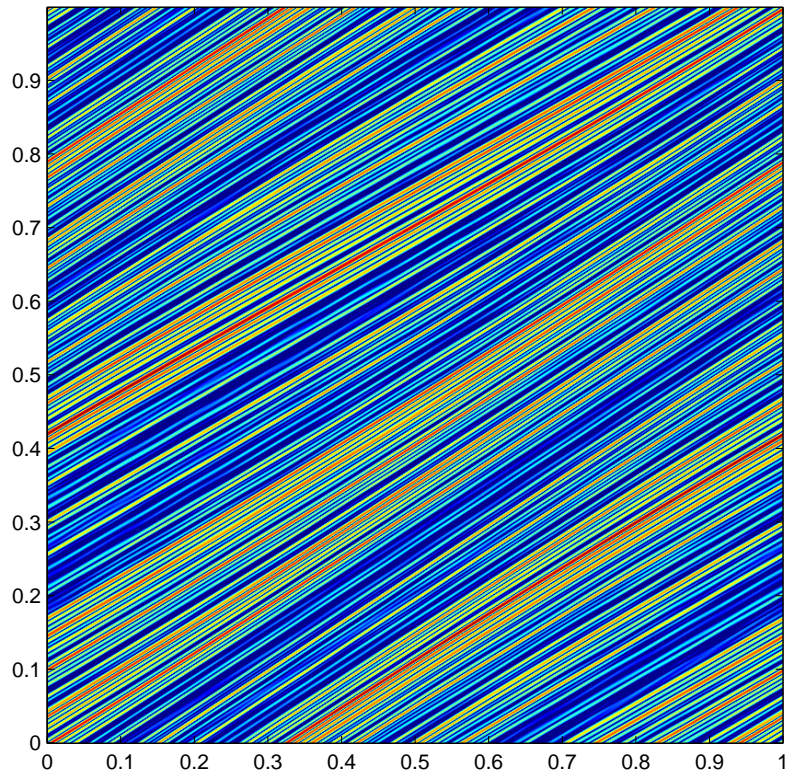


# Variance: 5 iterations for $\varepsilon = 0.3$ and $\kappa = 10^{-3}$

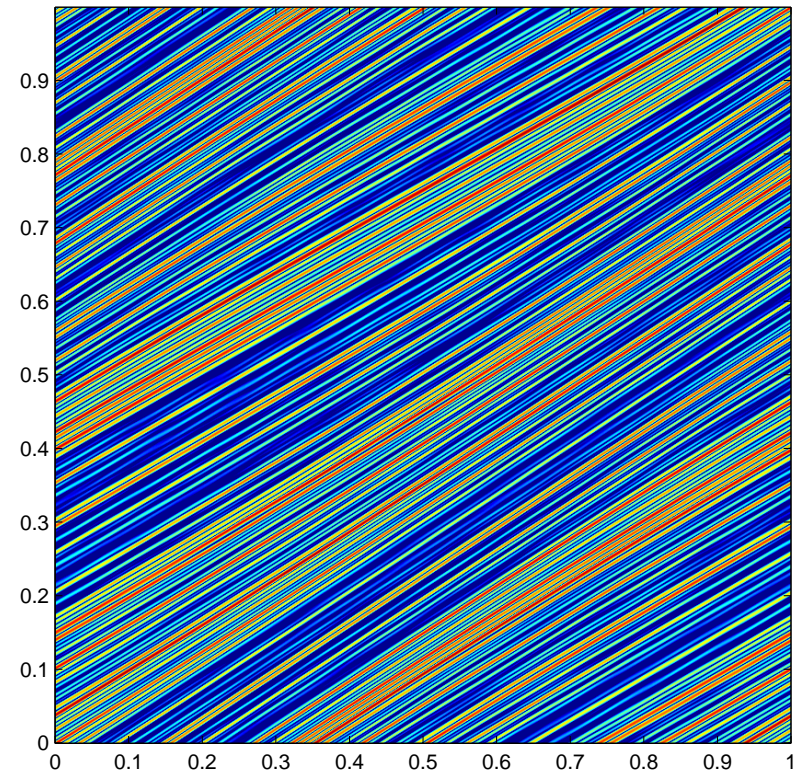


# Eigenfunction for $\varepsilon = 0.3$ and $\kappa = 10^{-3}$

(Renormalised by decay rate)



$i = 25$



$i = 30$

# Decay Rate

For small  $\varepsilon$ , the dominant Bessel function is  $J_1$ , so the decay factor  $\mu^2$  for the variance is given by

$$\mu = \left| \mathbb{T}_{(0\ 1), (0\ 1)} \right| = e^{-\kappa} J_1(\varepsilon) = \frac{1}{2}\varepsilon + \mathcal{O}(\kappa\varepsilon, \varepsilon^2).$$

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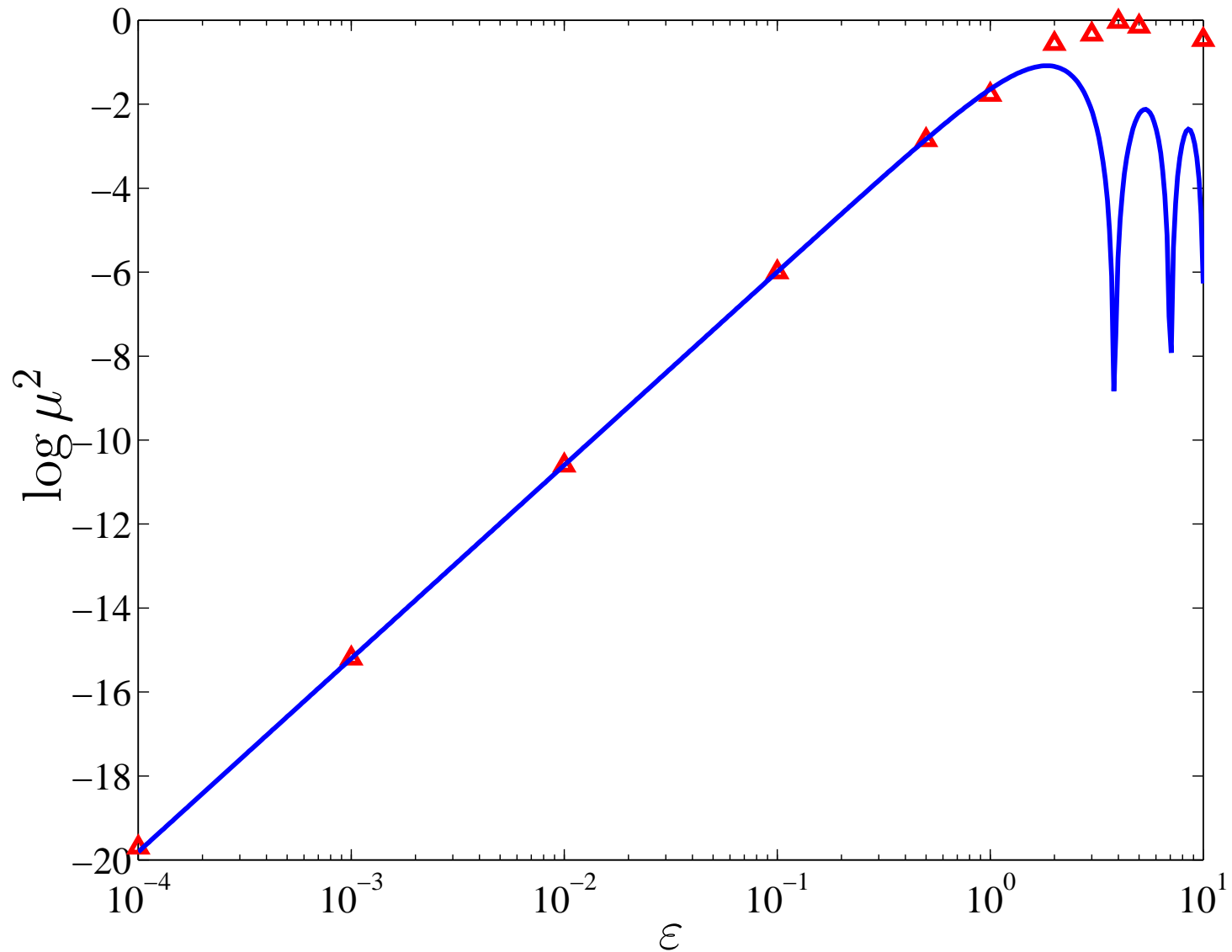
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In the baker's map the **discontinuity** imply a slow convergence of the Fourier modes. However, it is a one-dimensional problem.

# Decay Rate as $\kappa \rightarrow 0$



# Lagrangian Viewpoint

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- But must give same answer for a scalar quantity like the decay rate.

# Advection and Diffusion: Eulerian to Lagrangian

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Advection-diffusion (A-D) equation:

$$\partial_t \theta + \mathbf{v} \cdot \partial_{\mathbf{x}} \theta = \tilde{\kappa} \partial_{\mathbf{x}}^2 \theta.$$

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Transform A–D equation to Lagrangian coordinates,

$$\dot{\theta} = \partial_{\mathbf{X}} (\mathbb{D} \cdot \partial_{\mathbf{X}} \theta).$$

Anisotropic diffusion tensor, in terms of metric or Cauchy–Green strain tensor:

$$\mathbb{D} := \tilde{\kappa} g^{-1}; \quad g_{pq} := \sum_i \frac{\partial x^i}{\partial X^p} \frac{\partial x^i}{\partial X^q}.$$

# From Flow to Map

Velocity field doesn't enter the Lagrangian equation directly:  
regard the time dependence in  $\mathbb{D}$  as given by **map** rather than **flow**.

The solution of the A–D equation in Fourier space is then

$$\hat{\theta}_{\mathbf{k}}^{(i)} = \sum_{\ell} \exp(\mathcal{G}_{\mathbf{k}\ell}^{(i)}) \hat{\theta}_{\ell}^{(i-1)},$$

where  $i$  denotes the  $i$ th iterate of the map, and

$$\mathcal{G}_{\mathbf{k}\ell}^{(i)} = -4\pi^2 T \int_{\mathbb{T}^2} (\mathbf{k} \cdot \mathbb{D}^{(i)} \cdot \ell) e^{-2\pi i(\mathbf{k}-\ell) \cdot \mathbf{X}} d^2 X.$$

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This is an **exact result**, but the great difficulty lies in calculating the exponential of  $\mathcal{G}^{(i)}$ . We shall accomplish this **perturbatively**.

# Back to the Beginning

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$$\mathbb{M} = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}; \quad \boldsymbol{\phi}(\mathbf{x}) = \frac{\varepsilon}{2\pi} \begin{pmatrix} \sin 2\pi x_1 \\ \sin 2\pi x_1 \end{pmatrix};$$

The eigenvalues of  $\mathbb{M}$  are

$$\Lambda_u = \Lambda = \frac{1}{2}(3 + \sqrt{5}) = \cot^2 \theta, \quad \Lambda_s = \Lambda^{-1} = \frac{1}{2}(3 - \sqrt{5}) = \tan^2 \theta$$

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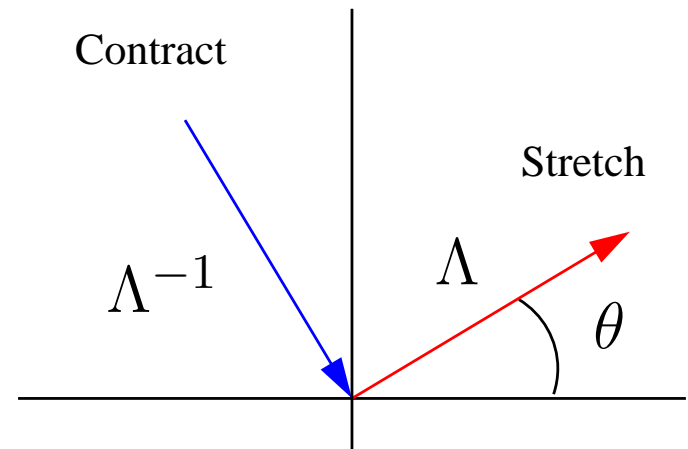
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# Coefficients of Expansion: Perturbation Theory

The coefficients of expansion and characteristic directions for the linear cat map are uniform in space. **Perturb off this.**

To leading order in  $\varepsilon$ , the coefficient of expansion is written as

$$\Lambda_\varepsilon^{(i)} = \Lambda^i (1 + \varepsilon \eta^{(i)})$$

where  $\Lambda$  is the coefficient of expansion for the unperturbed cat map; the perturbed eigenvectors are similarly written

$$\hat{\mathbf{u}}_\varepsilon^{(i)} = \hat{\mathbf{u}} + \varepsilon \zeta^{(i)} \hat{\mathbf{s}}, \quad \hat{\mathbf{s}}_\varepsilon^{(i)} = \hat{\mathbf{s}} - \varepsilon \zeta^{(i)} \hat{\mathbf{u}}.$$

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Simple application of matrix perturbation theory to Jacobian matrix of the map. The symmetrised Jacobian is the metric:

$$g_\varepsilon^{(i)} = [\Lambda_\varepsilon^{(i)}]^2 \hat{\mathbf{u}}_\varepsilon^{(i)} \hat{\mathbf{u}}_\varepsilon^{(i)} + [\Lambda_\varepsilon^{(i)}]^{-2} \hat{\mathbf{s}}_\varepsilon^{(i)} \hat{\mathbf{s}}_\varepsilon^{(i)}.$$

# Perturbation Results

$$\Lambda_\varepsilon^{(i)} = \Lambda^i (1 + \varepsilon \eta^{(i)}), \quad \hat{\mathbf{u}}_\varepsilon^{(i)} = \hat{\mathbf{u}} + \varepsilon \zeta^{(i)} \hat{\mathbf{s}},$$

$$\eta^{(i)} = \frac{1}{2} \sin 2\theta \sum_{j=0}^{i-1} \cos (2\pi (\mathbb{M}^j \cdot \mathbf{X})_1);$$

$$\zeta^{(i)} = \frac{1}{\Lambda^{2i} - \Lambda^{-2i}} (\zeta_+^{(i)} + \zeta_-^{(i)}),$$

$$\zeta_\pm^{(i)} = \frac{1}{2} (\cos 2\theta \mp 1) \sum_{j=0}^{i-1} \Lambda^{\pm 2(i-j)} \cos (2\pi (\mathbb{M}^j \cdot \mathbf{X})_1).$$

Observe that the perturbation to the eigenvectors **converges exponentially**, as required.

# Perturbed Metric Tensor

$$\mathbb{D}^{(i)} = \kappa [g_\varepsilon^{(i)}]^{-1}; \quad [g_\varepsilon^{(i)}]^{-1} = [\Lambda_\varepsilon^{(i)}]^{2i} \hat{\mathbf{s}}_\varepsilon^{(i)} \hat{\mathbf{s}}_\varepsilon^{(i)} + [\Lambda_\varepsilon^{(i)}]^{-2i} \hat{\mathbf{u}}_\varepsilon^{(i)} \hat{\mathbf{u}}_\varepsilon^{(i)}.$$

To leading order in  $\varepsilon$ , we have

$$\begin{aligned} [g_\varepsilon^{(i)}]^{-1} &= \Lambda^{2i} \hat{\mathbf{s}} \hat{\mathbf{s}} + \Lambda^{-2i} \hat{\mathbf{u}} \hat{\mathbf{u}} + 2\varepsilon \eta^{(i)} (\Lambda^{2i} \hat{\mathbf{s}} \hat{\mathbf{s}} - \Lambda^{-2i} \hat{\mathbf{u}} \hat{\mathbf{u}}) \\ &\quad - \varepsilon \zeta^{(i)} (\Lambda^{2i} - \Lambda^{-2i}) (\hat{\mathbf{u}} \hat{\mathbf{s}} + \hat{\mathbf{s}} \hat{\mathbf{u}}), \end{aligned}$$

where the only functions of  $\mathbf{X}$  are  $\eta^{(i)}$  and  $\zeta^{(i)}$ .

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To leading order in  $\varepsilon$ , we have

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where the only functions of  $\mathbf{X}$  are  $\eta^{(i)}$  and  $\zeta^{(i)}$ .

Recall the solution to the A–D equation:

$$\hat{\theta}_{\mathbf{k}}^{(i)} = \sum_{\ell} \exp(\mathcal{G}^{(i)})_{\mathbf{k}\ell} \hat{\theta}_{\ell}^{(i-1)}.$$

# The Exponent $\mathcal{G}^{(i)}$

---

$$\begin{aligned}\mathcal{G}_{\mathbf{k}\ell}^{(i)} &= -4\pi^2 T \int_{\mathbb{T}^2} (\mathbf{k} \cdot \mathbb{D}^{(i)} \cdot \ell) e^{-2\pi i(\mathbf{k}-\ell) \cdot \mathbf{X}} d^2 X \\ &= A_{\mathbf{k}\ell}^{(i)} + \varepsilon B_{\mathbf{k}\ell}^{(i)}\end{aligned}$$

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where

$$A_{\mathbf{k}\ell}^{(i)} = -\kappa \left( \Lambda^{2i} k_s^2 + \Lambda^{-2i} k_u^2 \right) \delta_{\mathbf{k}\ell}, \quad \kappa := 4\pi^2 \tilde{\kappa} T$$

$$\begin{aligned}B_{\mathbf{k}\ell}^{(i)} &= -\kappa \left( 2 \left( \Lambda^{2i} k_s \ell_s - \Lambda^{-2i} k_u \ell_u \right) \eta_{\mathbf{k}\ell}^{(i)} \right. \\ &\quad \left. - (k_u \ell_s + k_s \ell_u) \left( \zeta_+^{(i)} \mathbf{k}\ell + \zeta_-^{(i)} \mathbf{k}\ell \right) \right).\end{aligned}$$

with  $k_u := (\mathbf{k} \cdot \hat{\mathbf{u}})$ ,  $k_s := (\mathbf{k} \cdot \hat{\mathbf{s}})$ .

# The Exponent $\mathcal{G}^{(i)} = A^{(i)} + \varepsilon B^{(i)}$ (cont'd)

The diagonal part,  $A^{(i)}$ , inexorably leads to **superexponential** decay of variance, because it grows exponentially.

Upon making use of the Fourier-transformed  $\zeta^{(i)}$  and  $\eta^{(i)}$ , we find

$$B_{\mathbf{k}\ell}^{(i)} = -\frac{1}{2}\kappa \sum_{j=0}^{i-1} \mathcal{B}_{\mathbf{k}\ell}^{ij} \left( \delta_{\mathbf{k}, \ell + \hat{\mathbf{e}}_1 \cdot \mathbf{M}^j} + \delta_{\mathbf{k}, \ell - \hat{\mathbf{e}}_1 \cdot \mathbf{M}^j} \right)$$

$$\begin{aligned} \mathcal{B}_{\mathbf{k}\ell}^{ij} = & \sin 2\theta \left( \Lambda^{2i} k_s \ell_s - \Lambda^{-2i} k_u \ell_u \right) \\ & + (k_u \ell_s + k_s \ell_u) \left( \Lambda^{2(i-j)} \sin^2 \theta - \Lambda^{-2(i-j)} \cos^2 \theta \right). \end{aligned}$$

So  $B^{(i)}$  is **not diagonal** (it couples different modes to each other).

$\implies$  **Dispersive** in Fourier space.

# But can we Compute the Exponential, $\exp(\mathcal{G}^{(i)})$ ?

To leading order in  $\varepsilon$ , for  $A$  diagonal, we have

$$[\exp(A^{(i)} + \varepsilon B^{(i)})]_{\mathbf{k}\ell} = e^{A_{\mathbf{k}\mathbf{k}}^{(i)}} \delta_{\mathbf{k}\ell} + \varepsilon E_{\mathbf{k}\ell}^{(i)}; \quad E_{\mathbf{k}\ell}^{(i)} = B_{\mathbf{k}\ell}^{(i)} \frac{e^{A_{\mathbf{k}\mathbf{k}}^{(i)}} - e^{A_{\ell\ell}^{(i)}}}{A_{\mathbf{k}\mathbf{k}}^{(i)} - A_{\ell\ell}^{(i)}}.$$

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- For  $\varepsilon = 0$ , this is indeed what happens. But for a finite value of  $\varepsilon$ , the  $E$  term **breaks the diagonality of  $\mathcal{G}$** , so that given some initial set of wavevectors, the variance contained in those modes can be transferred elsewhere.

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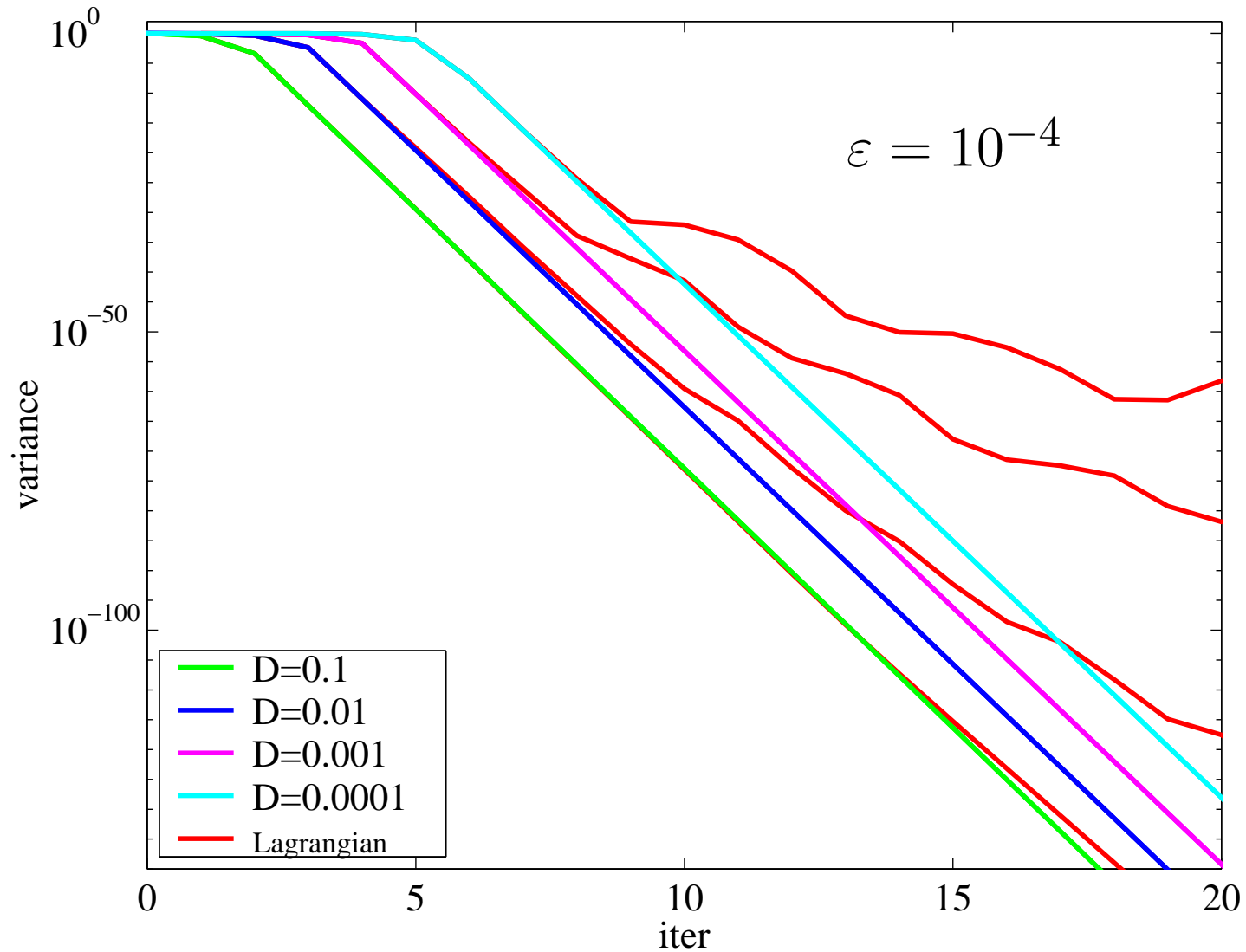
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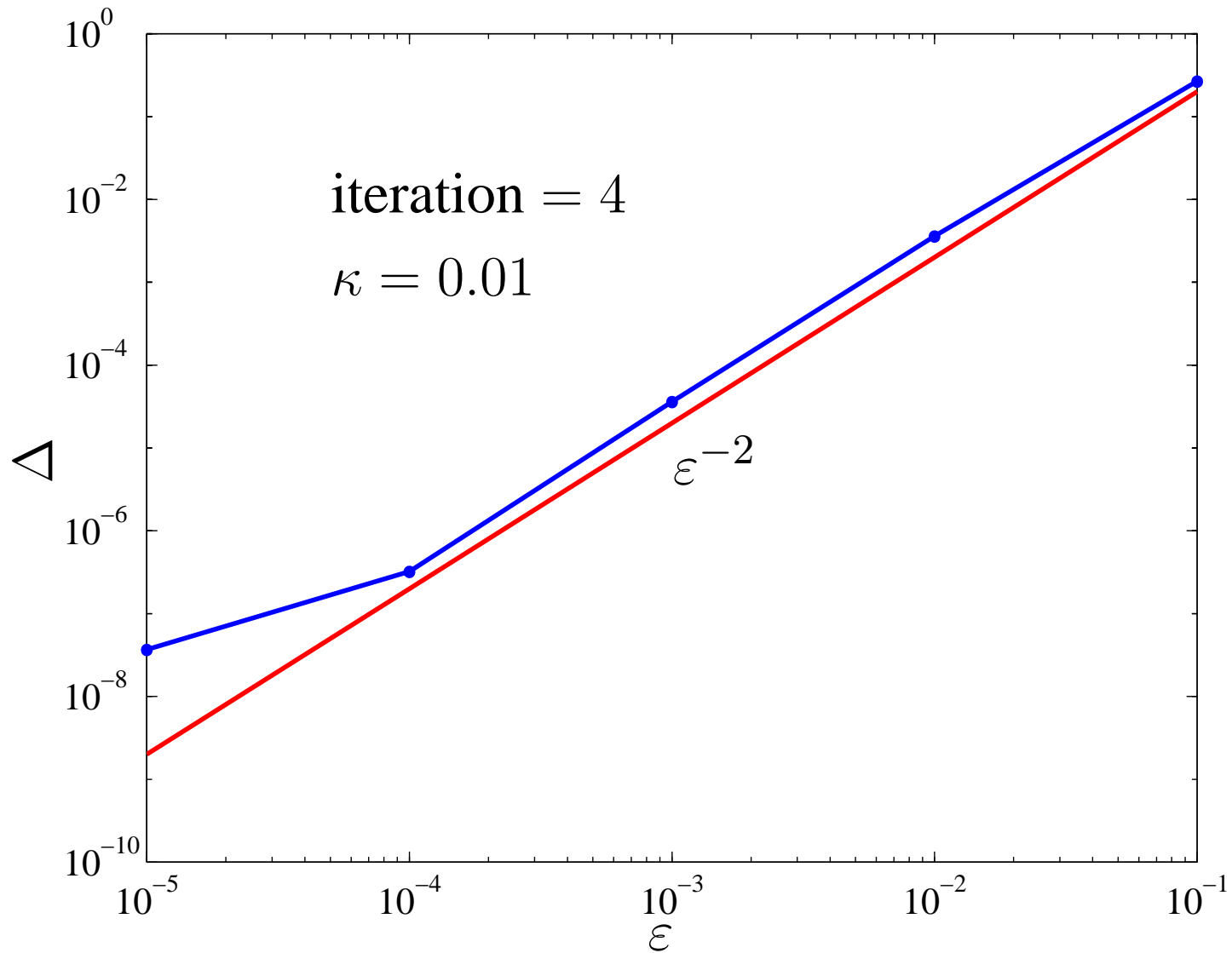
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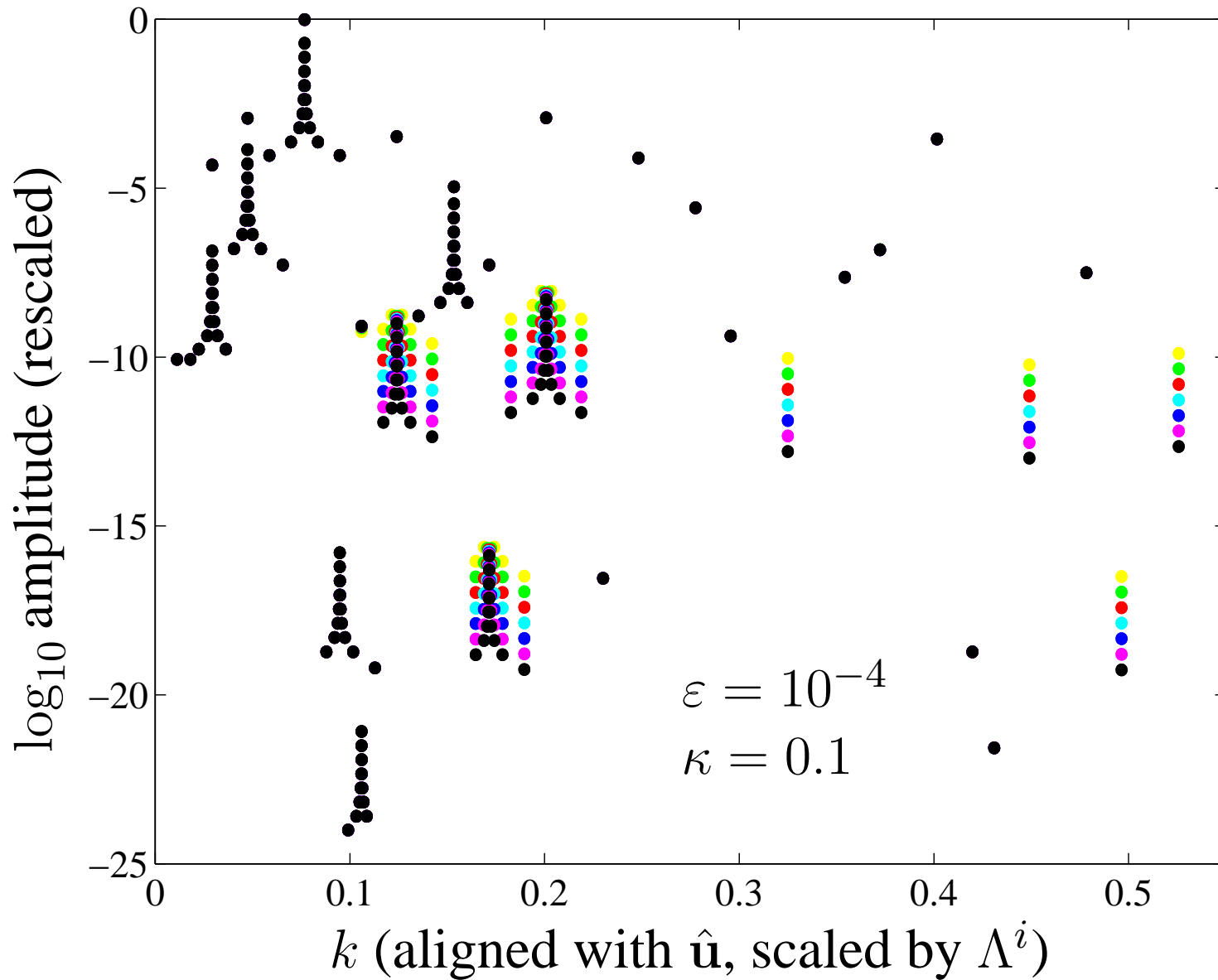
# Comparison: Eulerian and Lagrangian Views



# Convergence



# Rescaled Pattern for $i = 6, \dots, 12$



# Conclusions

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- Could the numerical economy be scaled to more difficult problems?
- Still some kinks to iron out!