Bound on Mixing Efficiency for the Advection-Diffusion Equation

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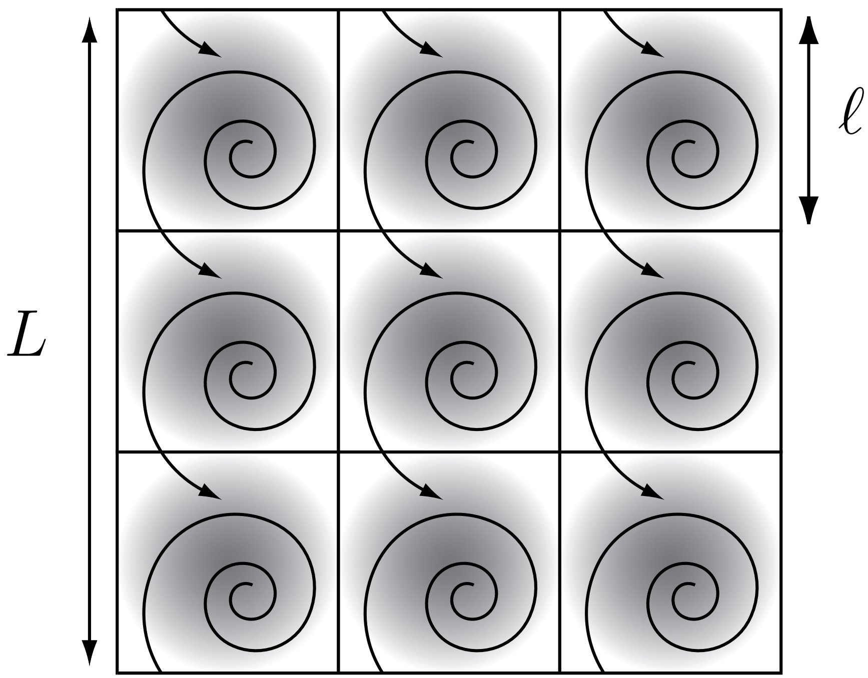
Summary

- Derive upper bound on the mixing efficiency for a passive scalar under the influence of advection and diffusion with a body source.
- Mixing efficiency measured in terms of an equivalent diffusivity.
- The precise value of the bound on the equivalent diffusivity depends only on the functional “shape” of both the source and the advecting field.
- Direct numerical simulations performed for a simple advecting flow to test the bounds.

The Setup



- Periodic system (2 or 3 dimensions)
- Stirring and source of scalar variance at scale ℓ
- System of size L
- Velocity field regarded as given, but could be time-dependent and even turbulent
- Source distribution could also be time-dependent

Advection–Diffusion Equation

$$\partial_t \theta + \boldsymbol{u} \cdot \nabla \theta = \kappa \, \Delta \theta + s \, ,$$

where κ is the molecular diffusivity and $s(\boldsymbol{x}, t)$ is a source function with zero spatial mean.
To characterise the fluctuations in θ, we use the variance,

$$\Theta^2 \equiv \left\langle L^{-d} \, \|\theta\|^2_{L^2(\mathbb{T}^d)} \right\rangle$$

The angle brackets $\langle \cdot \rangle$ denote a long-time average, and $\|\cdot\|_{L^2(\mathbb{T}^d)}$ is the L^2 norm on $\mathbb{T}^d$. Decompose s and $\boldsymbol{u}$ as

$$s(\boldsymbol{x}, t) = S \, \Phi(\boldsymbol{x}/\ell, t/\tau), \quad \left\langle L^{-d} \, \|\Phi\|^2_{L^2(\mathbb{T}^d)} \right\rangle = 1,$$

$$\boldsymbol{u}(\boldsymbol{x}, t) = U \, \boldsymbol{\Upsilon}(\boldsymbol{x}/\ell, t/\tau), \quad \left\langle L^{-d} \, \|\boldsymbol{\Upsilon}\|^2_{L^2(\mathbb{T}^d)} \right\rangle = 1.$$

The Bounds

Introduce an arbitrary function Ψ that satisfies

$$\left\langle \int_{\mathbb{I}^d} \Psi(\boldsymbol{y})\, \Phi(\boldsymbol{y})\, \mathrm{d}^d y \right\rangle = 1, \qquad \boldsymbol{y} = \boldsymbol{x}/\ell.$$

After some manipulation, obtain the bound

$$S \le \frac{U\Theta}{\ell} \left(c_1 + \mathrm{Pe}^{-1}\, c_2 \right),$$

where $c_1 \equiv \left\langle \|\boldsymbol{\Upsilon} \cdot \nabla_{\boldsymbol{y}} \Psi\|^2_{L^2(\mathbb{I}^d)} \right\rangle^{1/2}, \quad c_2 \equiv \left\langle \|\Delta_{\boldsymbol{y}} \Psi\|^2_{L^2(\mathbb{I}^d)} \right\rangle^{1/2}.$

are dimensionless constants, independent of Pe and Θ.
c_1 depends explicitly on the stirring shape-function $\boldsymbol{\Upsilon}$ and implicitly on the source shape function through Ψ.

Bounds on Equivalent Diffusivity

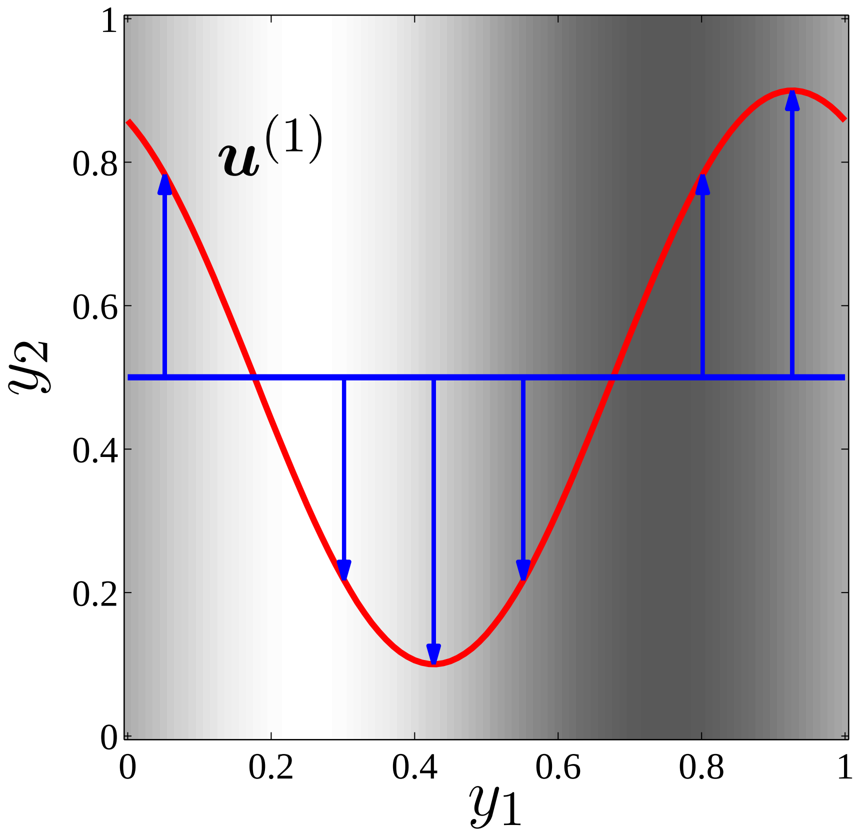
Express bound in terms of an equivalent diffusivity:

$$\kappa_{\mathrm{eq}} \equiv \frac{S\ell^2}{\Theta} \leq c_1\, U\ell + c_2\, \kappa,$$

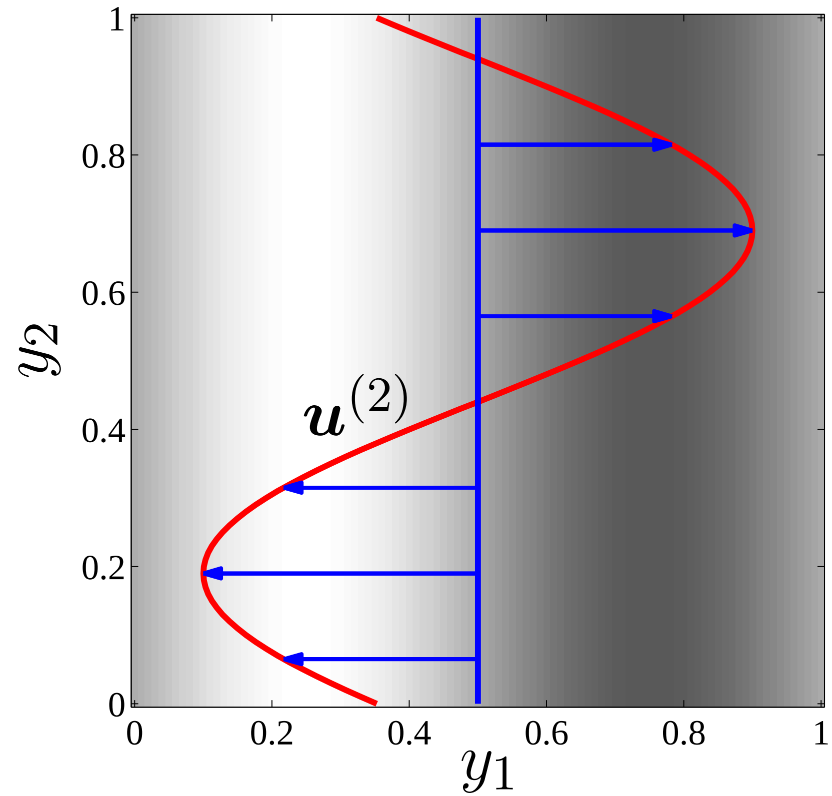
- The equivalent diffusivity compares the source amplitude (S) to the steady-state fluctuations in the concentration field (Θ).
- A high Péclet number ($\mathrm{Pe} \equiv U\ell/\kappa$) mixing device should operate with as high a κ_{eq} as possible compared to κ.
- $\kappa_{\mathrm{eq}} = \kappa$ for $U = 0$, which is the purely diffusive case (after a trivial normalisation, not included above).
- The scaling $U\ell$ is often used as an estimate for turbulent diffusivity, but here we have an explicit prefactor that depends on the stirring and source distribution.

Random Sine Flow

Alternating horizontal and vertical sine shear flows, with randomized phase. Source distribution is fixed.



$0 < t < \tau/2$



$\tau/2 < t < \tau$

Comparison with Numerical Results