The structure of random braids

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Tangled magnetic fields

The "vision" for solar flux tubes

magnetic field

sunspot

"braided"

The magnetic field lines become "braided" due to MHD frozen-in condition + turbulence
Random braids

So far studies focus on given braid: say, the pistail braid [Wilmot-Smith, Hornig, Yeats, ...]
But how do we generate an "appropriate" random braid?

Previous work: Berger (invariants), Sumners (random knots)
Nechaev ("entangled random walks") => Cayley graphs
[more refs deleted]

I will summarize some earlier work and point to some difficulties.
The braid group $B_n$

A braid is a set of $n$ strands with fixed endpoints.

Two braids are equal if they can be "deformed" into each other, whilst holding ends fixed. "ambient isotopy".

Braids form a group: (for fixed $n$)

Identity:

This is associative, and for every braid there is an inverse that "disentangles" the braid.
Braid generators

\[ \sigma_1^{-1} \sigma_3 \sigma_4 \sigma_3^{-1} \]

This braid can be written

\[ \sigma_1 \sigma_3 \sigma_4 \sigma_3^{-1} \]

\( \{ \sigma_1, \sigma_2, \ldots, \sigma_{n-1} \} \) are generators of \( B_n \)

They satisfy relations:

\[ \sigma_i \sigma_j = \sigma_j \sigma_i, \quad |i-j| > 1 \]

\[ \sigma_i \sigma_j \sigma_i = \sigma_j \sigma_i \sigma_j, \quad |i-j| = 1 \]

Artin proved that these are the only relations that arise from physical braids.
Cayley graph

A convenient graphical representation of groups is as a graph:

The Cayley graph for $B_3$ might start out like this, but there are "shortcuts" (loops) due to braid relations.

Now we can define a random walk on this graph by choosing a random direction to move to from each vertex.

$\tau = \text{random word} = \sigma_1 \sigma_2^{-1} \sigma_1 \sigma_2 \sigma_1^{-1} \sigma_2 \sigma_1 \sigma_2^{-1} \ldots$ (say)

Lots of interesting questions! (recurrence, distance, etc...)
The simplest Cayley graph

For $B_2$, only one generator

\[ \ldots \sigma_2 \sigma_1^{-1} \sigma_3 \sigma_1^{-1} \sigma_2 \sigma_1^{-1} \sigma_2^{-1} \sigma_3 \sigma_1 \sigma_2 \sigma_1^{-1} \sigma_3 \sigma_1^{-1} \sigma_2 \sigma_1^{-1} \sigma_2^{-1} \sigma_3 \]

The exponent gives the winding number of one strand around the other.

If our random walk moves to the left/right with probability $p/(1-p)$, expect net winding number $m$ after $N$ steps to have probability

\[ P(m) = {N \choose k_m} p^{k_m} (1-p)^{N-k_m} \]

\[ k_m = \frac{1}{2} (m+N) \]

(m+N even!)

Mean $N(p-\frac{1}{2})$, variance $Np(1-p) \Rightarrow$ Gaussian for large $N$
Winding number of two Brownian processes

Consider now two Brownian processes on the plane, diffusivity $D$. $\langle z_i^3 \rangle \sim 2Dt$ (Think of two stochastic field lines)

Picture as a braid:

What is the distribution of the winding angle $\theta$, after a large time $t$?

Same as Cayley graph random walk?
Winding around the origin

Let \( \mathbf{X}(t) = \mathbf{z}_1(t) - \mathbf{z}_2(t) \rightarrow \) Brownian with diffusivity 2D

\[
\langle X^2 \rangle = \langle z_1^2 \rangle + \langle z_2^2 \rangle = 2(2Dt)
\]

So now the question is: how many times does \( \mathbf{X} \) wind around the origin?

Classic result of Spitzer (1958):

\[
p(x) \sim \frac{1}{\pi} \frac{1}{1 + x^2}
\]

\[
x = \frac{2\Theta}{\log(4Dt/r_0^2)} \quad \frac{Dt}{r_0^2} \gg 1
\]

Cauchy–Lorentz \( \rightarrow \) not Gaussian!
How do we show this?

Brownian process $\Rightarrow$ heat equation

Solve in wedge of angle $2\alpha$:

$$\frac{\partial P}{\partial t} = D \nabla^2 P \quad P_0 = \delta(z-z_0)$$

with Neumann conditions (conservation of probability)

Compute Green's function (e.g. Carslaw & Jaeger)
then take $\alpha \to \infty$! $\rightarrow$ multiple Riemann sheets

large- $t$ asymptotics then give Cauchy distribution.
The problem with the tails

In practice, we never see this. The large winding angles predicted by Spitzer are a symptom of the scale-free Brownian process. It can wind very fast around origin.

Any "regularization" (random walk, length scale, curvature-limited...) gives

$$p(\alpha) \sim \frac{1}{2} \text{sech}\left(\frac{\pi\alpha}{2}\right)$$

$\leftarrow$ exponential tails (still not Gaussian)

[One way to get this: Take out dish around origin]
[ Pitman & Yor '86, Berger '87, Drozd & Kardar '96, Grodberg & Frisch '03]
Scaling of $x$ with $\log t$

The scaling $x \sim 2\theta / \log(4Dt/r_0^2)$ arises because the Brownian process wanders away from the origin $\rightarrow$ PDF stops changing.

Scaling argument:

$$\frac{dr}{d\theta} = r f(\theta) \quad \text{since $r$ is the only length scale for $Dt \gg r_0^2$}$$

$\rightarrow$ indep. of $\theta$ by isotropy

So $d\theta \sim \frac{dr}{r} \sim \frac{dt}{t} \sim d\log t \quad \text{since $r \sim t^{1/2}$}$

[See Fisher et al. 1984, Drossel & Kärger 1996]
Closed domain

Now all this was for a Brownian motion on the plane. In a confined geometry, the process "starts over" when it reflects. [Markovian]

So pieces of random walk of duration

\[ \frac{R^2}{D} \]

diffusion time across disk are uncorrelated, and each piece has a Spitzer distribution.

Convolving the \( t/ (R^2/D) \) distributions then gives

\[ \rho \sim \frac{\Theta}{\sqrt{t}} \]

rather than \( \Theta/ \log t \)

[ Dressel & Kardar 2003]

Confinement leads to many more turns
Blinking vortex simulations

Winding number for blinking vortex pair in a disk (Aref, 1984):

The red curve is the sech distribution, with fitted diffusivity.
The moral so far

Maybe I’ve convinced you that the random walk on a Cayley graph (which suggest Gaussian winding angle) is hard to reconcile with the twisting of two random walkers. (Not sure how to do it...)

At least in our two-walker example the probability of winding in one direction was the same as the other. $P(\tau_i) = P(\bar{\tau}_i)$

But more generally, for $n$ random walks, are all braid generators $\{\tau_i, \ldots, \tau_{n-1}\}$ equally likely?
Random walkers in an arbitrary domain

When projecting along a fixed line, each "crossing" of two walkers corresponds to a generator $\sigma^\pm$.

Since each walker is independent and uniformly distributed, the probability of observing a crossing depends on width.

[See JLT 2005, 2010]
Where do crossings occur?

The probability of a walker being in \([x, x+dx]\) is

\[ p(x) \, dx = \frac{1}{A} y(x) \, dx \]

(area)

\[ \int_0^1 p(x) \, dx = 1 \]

The probability of a crossing occurring at \(x\) is

\[ P(\text{crossing at } x) = \frac{p^2(x)}{\int_0^1 p^2(x) \, dx} \]

[two particles at \(x\) is a crossing]

[This is in the small step-size limit]
Which generator?

Once we have a crossing, we can ask which generator it corresponds to.

This depends on the ordering of the particles.

Find: [See Sarah Turnas's thesis]

\[ P(\sigma_h^{\pm 1}) = (n-2) \int_0^1 p(x) P^{h-1}(x' < x) P^{n-h-1}(x' > x) \, dx \]

For a square domain, \( p(x) = 1 \) and

\[ P(\sigma_h^{\pm 1}) = \frac{1}{n-1} \]

All generators projected along the \( x \)-axis occur with equal probability.
Distribution of generators in a disk

PDF of $|\text{generators}|$ for a random walk in the disk
Conclusions

- Can generate braids by ‘randomly picking generators’ (random walk on Cayley graph), but not clear what physical process that corresponds to;
- Brownian motions have Cauchy-distributed winding angles;
- Random walks have sech-distributed winding angles, as does a simple chaotic flow;
- The braids created by random walks depend on the shape of the domain!
- Topological entropy of random braids?
- I have a postdoctoral position available to work on braids and coherent structures! See www.math.wisc.edu/~jeanluc.
References


