
Turbulent Polymers

Stretching and Degradation

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Drag Reduction

Experimental facts:

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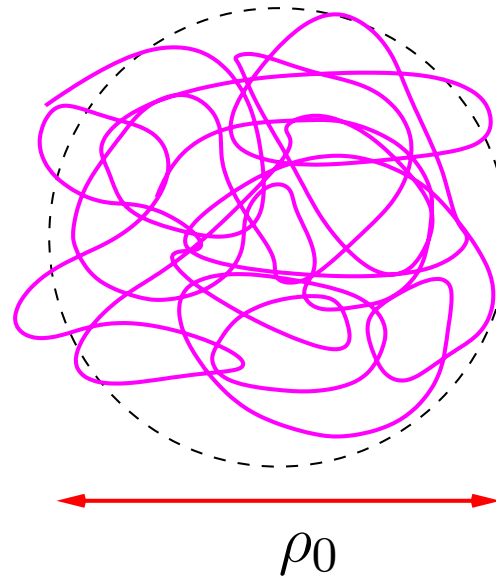
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- **Stretched state**: crucial for drag reduction. Causes polymers to break (**degradation**).
- Use simple turbulence model to investigate.

Polymers

Polymers are long chains of molecules. Random walk at rest.

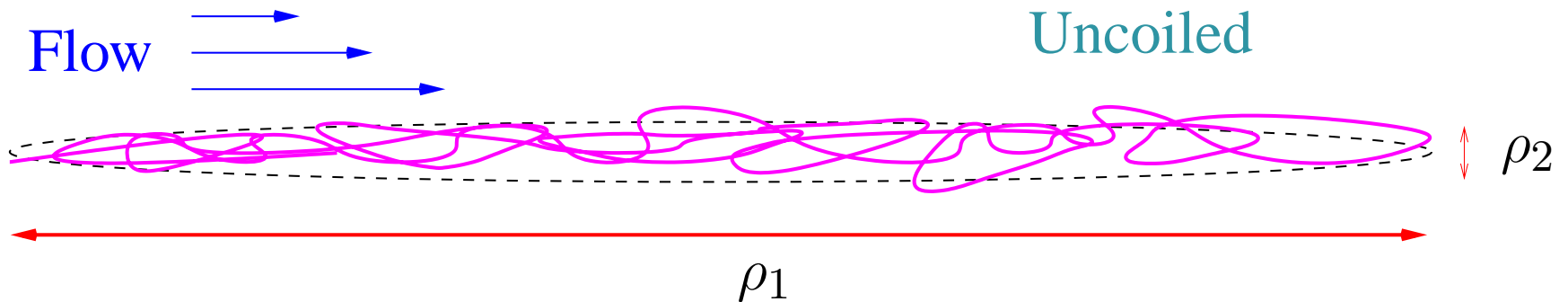
No flow

Coiled



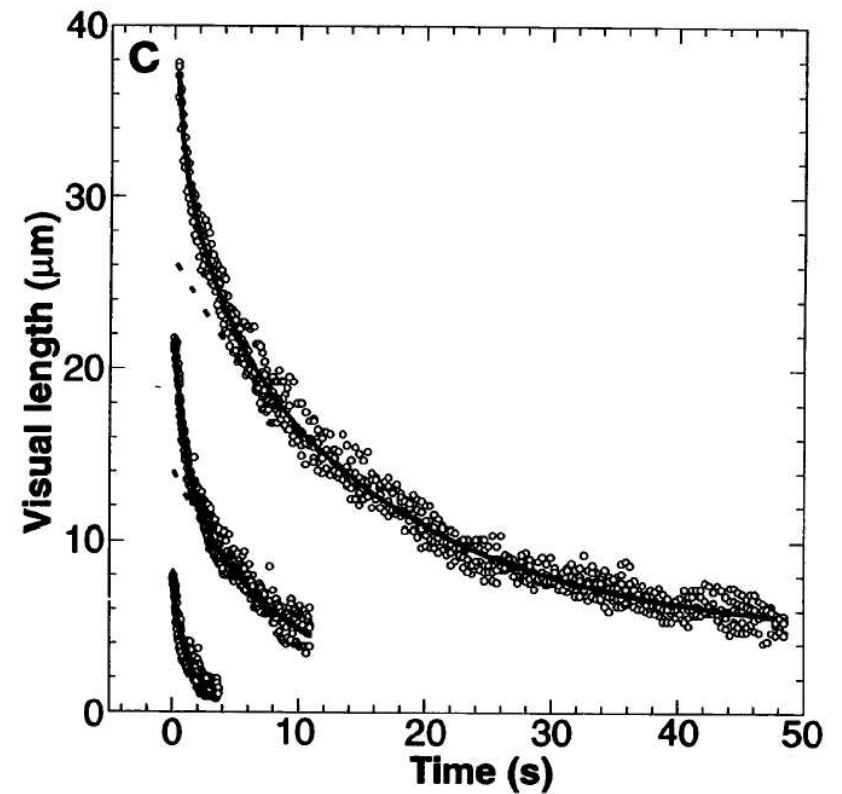
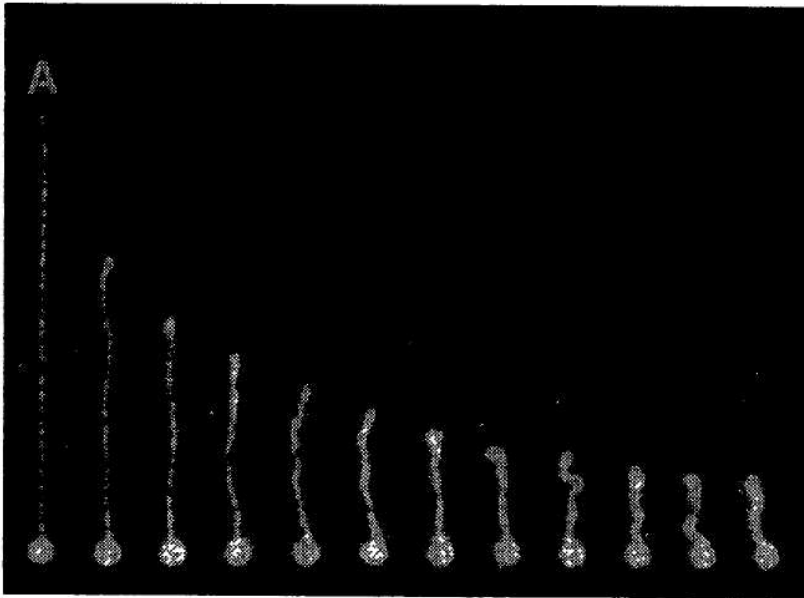
Flow

Uncoiled



Relaxation of Polymers

Pretty close to exponential...



From Perkins *et al.*, *Nature* (1994)

Constitutive Models

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- Viscoelastic:

$$T_{ij} = \int_{-\infty}^t G(t - t') \dot{\gamma}_{ij}(t') dt'$$

Maxwell Model

Exponentially-decaying memory:

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Not good as a fluid relation. Remedied by introducing a frame-independent (**Oldroyd**) derivative

$$\dot{\mathbb{T}} \quad \Longrightarrow \quad \overset{\nabla}{\mathbb{T}} \equiv \frac{\partial \mathbb{T}}{\partial t} + \mathbf{u} \cdot \nabla \mathbb{T} - (\mathbb{T} \cdot \nabla \mathbf{u} + (\nabla \mathbf{u})^T \cdot \mathbb{T})$$

The Equations of Motion

Couple stress to Navier–Stokes for an incompressible fluid:

$$\frac{\partial \mathbf{u}}{\partial t} + \mathbf{u} \cdot \nabla \mathbf{u} + \nabla p = \nu \nabla^2 \mathbf{u} + \frac{s}{\tau} \nabla \cdot \mathbb{A};$$
$$\overset{\nabla}{\mathbb{A}} = -\frac{2}{\tau} (\mathbb{A} - \rho_0^2 \mathbb{I}); \quad \nabla \cdot \mathbf{u} = 0,$$

where \mathbb{A} is equal to \mathbb{T} up to constants, and can be regarded as the local **deformation** of the polymers, with $\mathbb{A} = \rho_0^2 \mathbb{I}$ at rest.

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Can be derived from a kinetic model of “**Hookean dumbbells**.”

More generally: allow **nonlinear saturation** of the length of polymers (**FENE**-type models)

$$\overset{\nabla}{\mathbb{A}} = -\frac{2}{\tau} (f(A) \mathbb{A} - \rho_0^2 \mathbb{I}).$$

Evolution of the Principal Axes

The polymer **conformation tensor** \mathbb{A} can be diagonalized, with orthonormal eigenvectors \mathbf{e}_α and eigenvalues $(\rho_\alpha)^2$ that evolve according to

$$\frac{d\rho_\alpha}{dt} = \lambda_\alpha \rho_\alpha - \frac{1}{\tau} (f(\|\rho\|^2)\rho_\alpha - \rho_0^2/\rho_\alpha),$$

$$\lambda_\alpha(t, \mathbf{x}) \equiv \mathbf{e}_\alpha \cdot \nabla \mathbf{u} \cdot \mathbf{e}_\alpha, \quad d/dt = \partial/\partial t + \mathbf{u} \cdot \nabla.$$

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The ρ_α are the **lengths of the principal axes** of the ellipsoid delineating the deformation of the polymer.

If the **correlation time** t_{corr} is long enough, the polymers tend to align with the dominant stretching direction, so we consider only the major axis:

$$\frac{d\rho}{dt} = \lambda \rho - \frac{1}{\tau} (f(\rho^2)\rho - \rho_0^2/\rho)$$

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In that case, model the velocity field as a **Gaussian random variable** representing a smooth straining field $\lambda(t)$ that changes rapidly; λ satisfies

$$\langle \lambda(t)\lambda(t') \rangle - \bar{\lambda}^2 = \delta(t - t') \Delta; \quad \langle \lambda(t) \rangle = \bar{\lambda},$$

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The variable $\lambda(t)$ is **δ -correlated** in time, which means that it forgets about its previous state immediately. It has mean $\bar{\lambda}$ and standard deviation Δ .

This “slightly” artificial situation has great analytical advantages.

Fokker–Planck Equation

Because the distribution of $\lambda(t)$ is Gaussian and δ -correlated, can obtain a **Fokker–Planck equation** for the distribution $\mathcal{P}(t, \rho)$ of the major axis:

$$\mathcal{Z}(t; \mu) = \langle \exp(i \mu \rho) \rangle$$

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Inverse Fourier transformation of \mathcal{Z} with respect to μ then gives the equation of motion for $\mathcal{P}(t, \rho)$, the **PDF of ρ** , (**Chertkov**, 2000)

$$\partial_t \mathcal{P} = \frac{1}{2} \Delta \partial_\rho \rho \partial_\rho \rho \mathcal{P} - \bar{\lambda} \partial_\rho \rho \mathcal{P} + \frac{1}{\tau} \partial_\rho (f(\rho^2) \rho - \rho_0^2 / \rho) \mathcal{P}$$

Stationary Distribution: Coiled State

Natural thing to do is to look for **stationary solutions** of the F–P equation. Assume first that the polymers are uncoiled. Neglect nonlinear relaxation: $f = 1$ (Hookean springs).

$$\mathcal{P}_c(\rho) \sim \rho^{-1-2(\xi-\zeta)} \exp(-\xi \rho_0^2 / \rho^2)$$

where ρ is normalized by ρ_0 , and

$$\xi \equiv 1/\Delta\tau, \quad \zeta \equiv \bar{\lambda}/\Delta$$

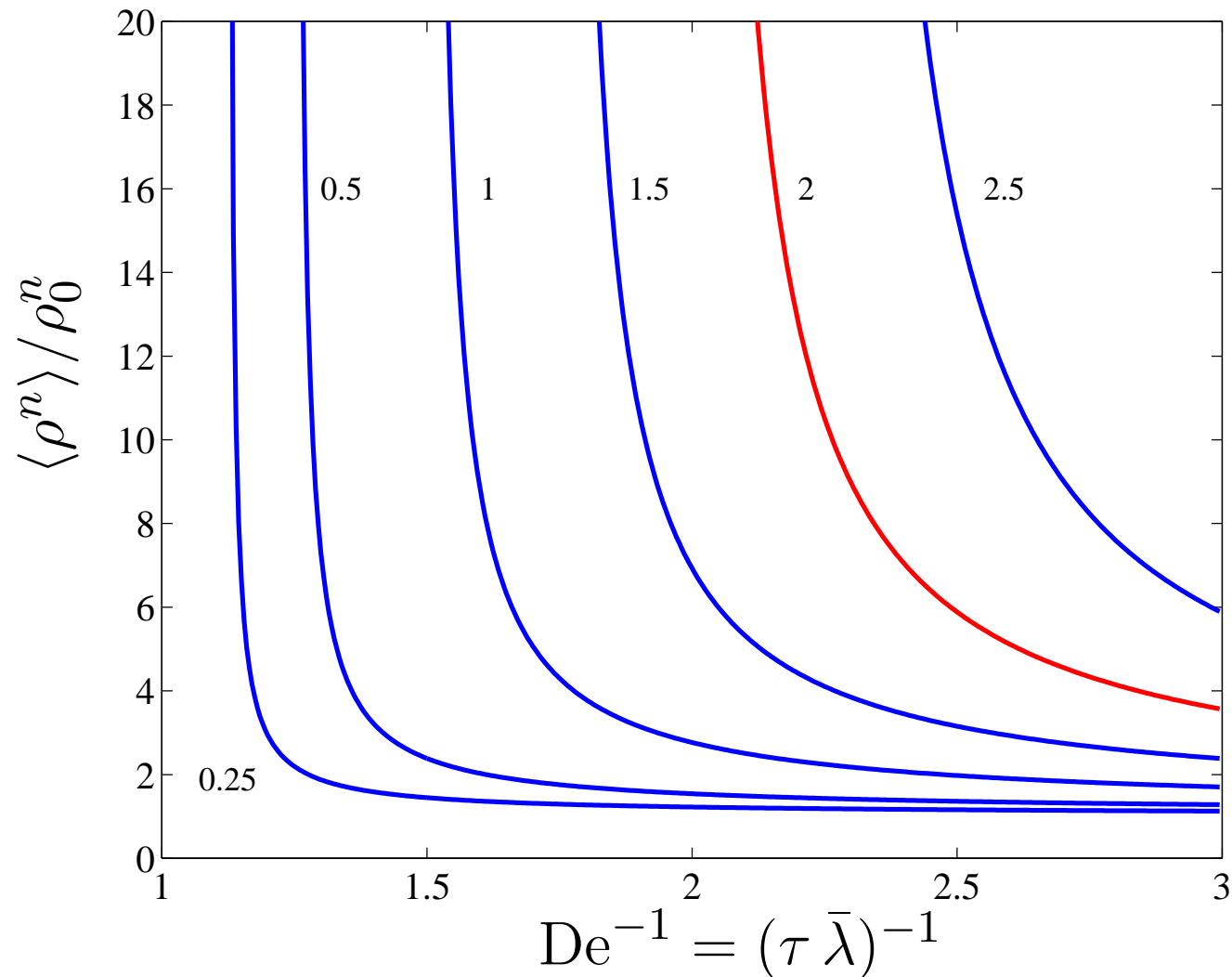
are dimensionless numbers.

$\zeta/\xi = \text{De}$, the **Deborah number**, the ratio of the **polymer relaxation timescale**, τ , over the **advection timescale**, $1/\bar{\lambda}$.

Large De \implies polymers more affected.

Moments of the Distribution

Below the coil-stretch transition at $De = 1$ (with $\bar{\lambda}/\Delta = 1$).



Stationary Distribution: Stretched State

Assuming a **FENE**-type (**F**inite **E**xtension **N**onlinear **E**lastic) model which limits the length of the polymers to ρ_m ,

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Can find equilibrium distribution

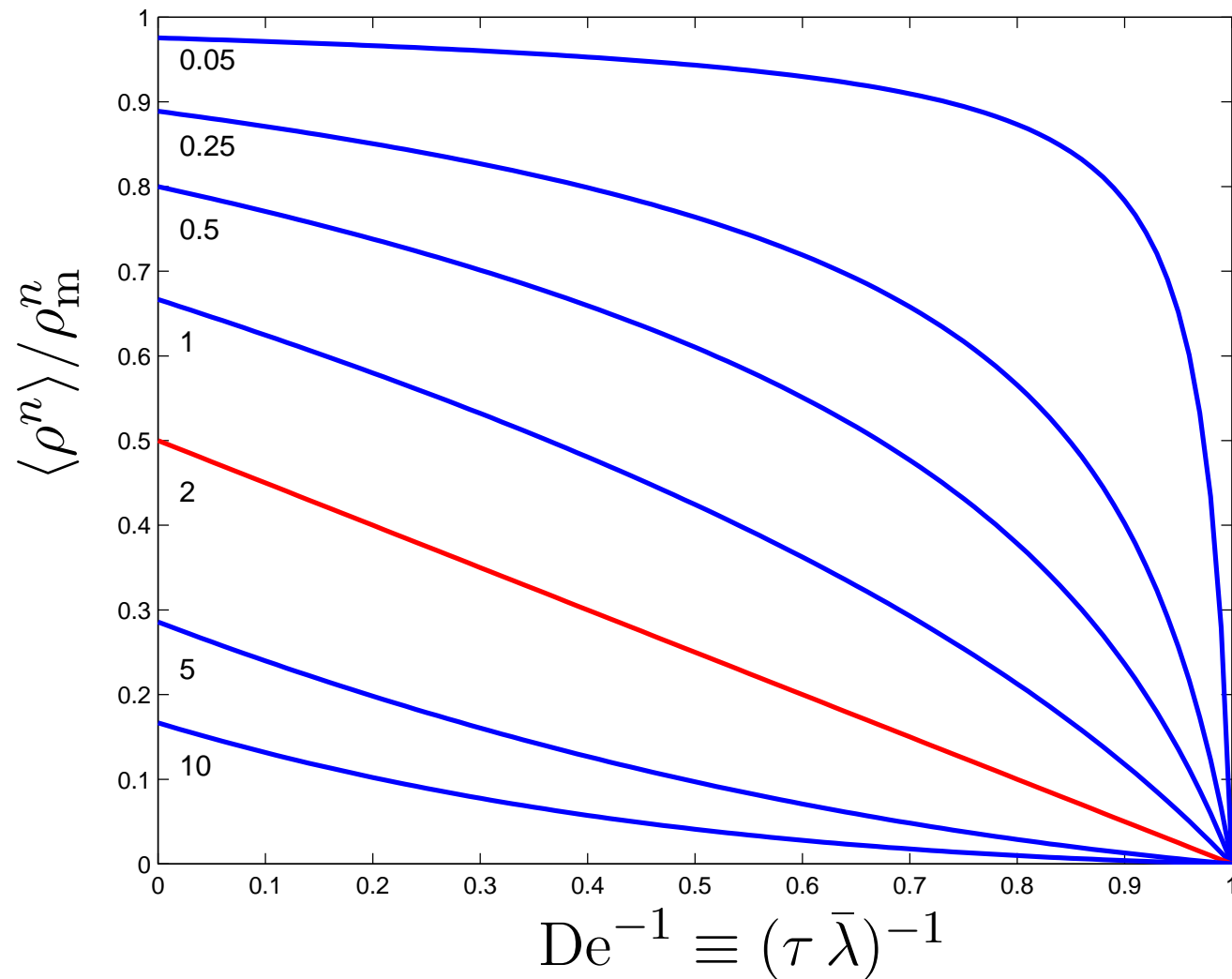
$$\mathcal{P}_s(\rho) \sim \rho^{-1+2(\bar{\lambda}/\Delta-\xi)} (1 - \rho^2)^\xi$$

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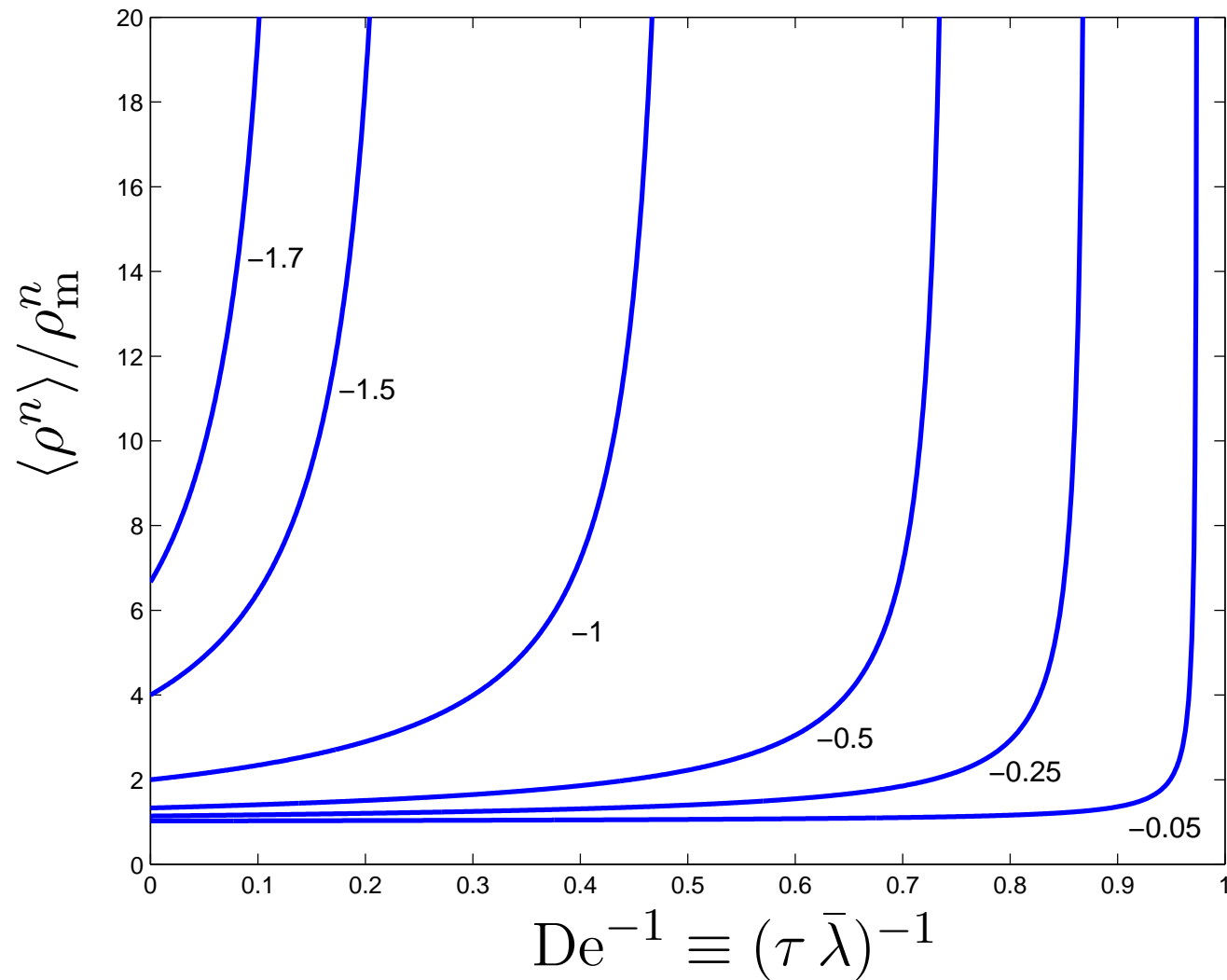
\implies Cutoff in PDF for $\rho > \rho_m$.

Moments of the Distribution

Coil-stretch transition at $De = 1$ (with $\bar{\lambda}/\Delta = 1$).



Negative moments



Breaking Point

The PDF of length can also be used to estimate the **fraction of polymers that break**.

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To that critical tension corresponds a critical length, ρ_c , obtained by solving

$$F_c = \frac{1}{\tau} f(\rho_c^2)\rho_c = \frac{1}{\tau} \frac{\rho_m^2}{\rho_m^2 - \rho_c^2} \rho_c,$$

$$\implies \rho_c = \frac{\rho_m^2}{2\tau F_c} \left(\sqrt{1 + (2\tau F_c/\rho_m)^2} - 1 \right).$$

The Survivors

The fraction of polymers that **survive** is equal to the **fraction shorter than ρ_c** , obtained by integrating the PDF from 0 to ρ_c ,

$$\text{Prob}(\rho < \rho_c) = \frac{B(\rho_c^2/\rho_m^2; \zeta - \xi, \xi + 1)}{B(\zeta - \xi, \xi + 1)},$$

where

$$B(a, b) \equiv \Gamma(a)\Gamma(b)/\Gamma(a + b)$$

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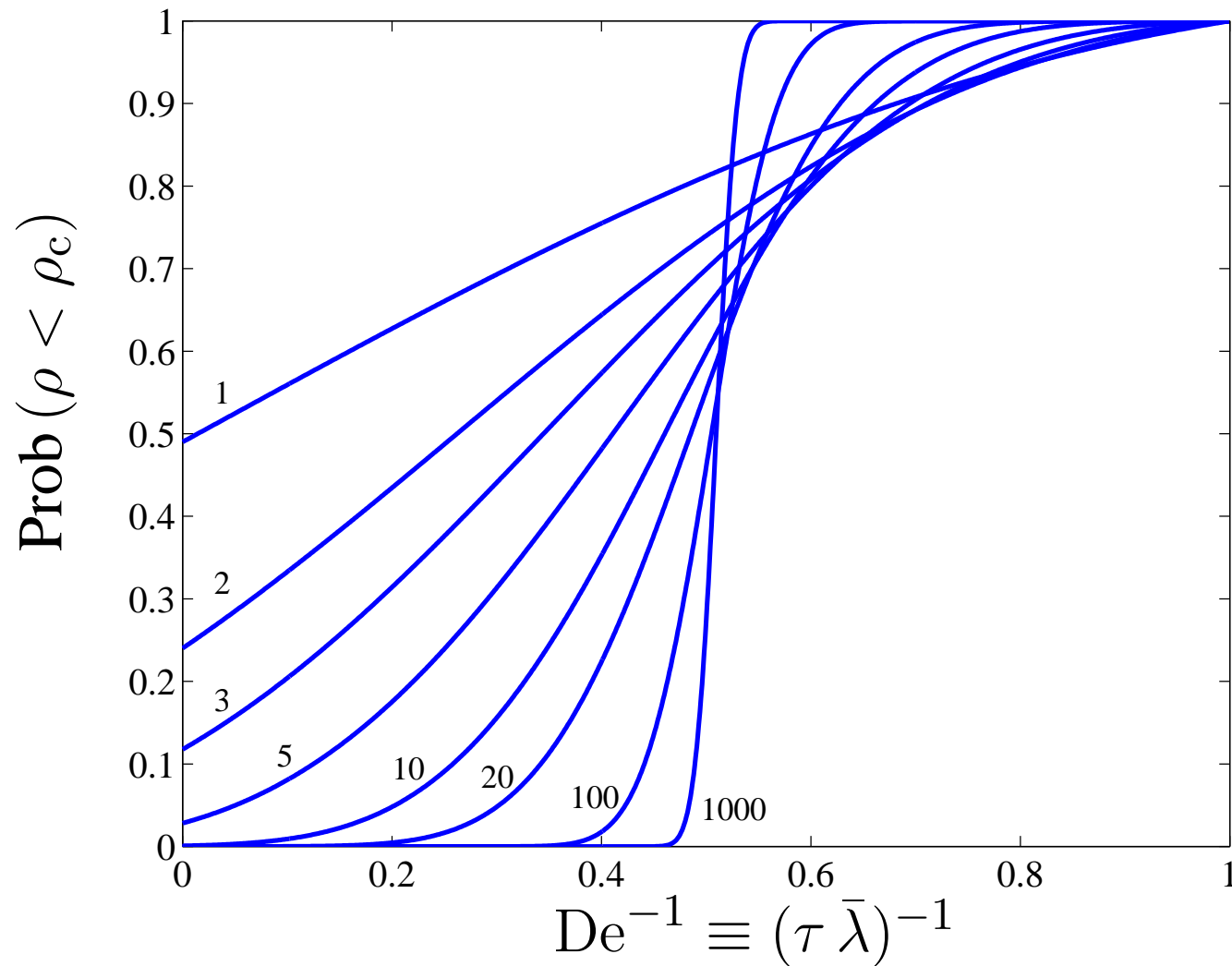
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Since $B(1; a, b) = B(a, b)$, for $\rho_c = \rho_m$ none of the polymers break.

Surviving Fraction as a function of De^{-1}

For different values of $\zeta \equiv \bar{\lambda}/\Delta$ with $\rho_c/\rho_m = 0.7$.



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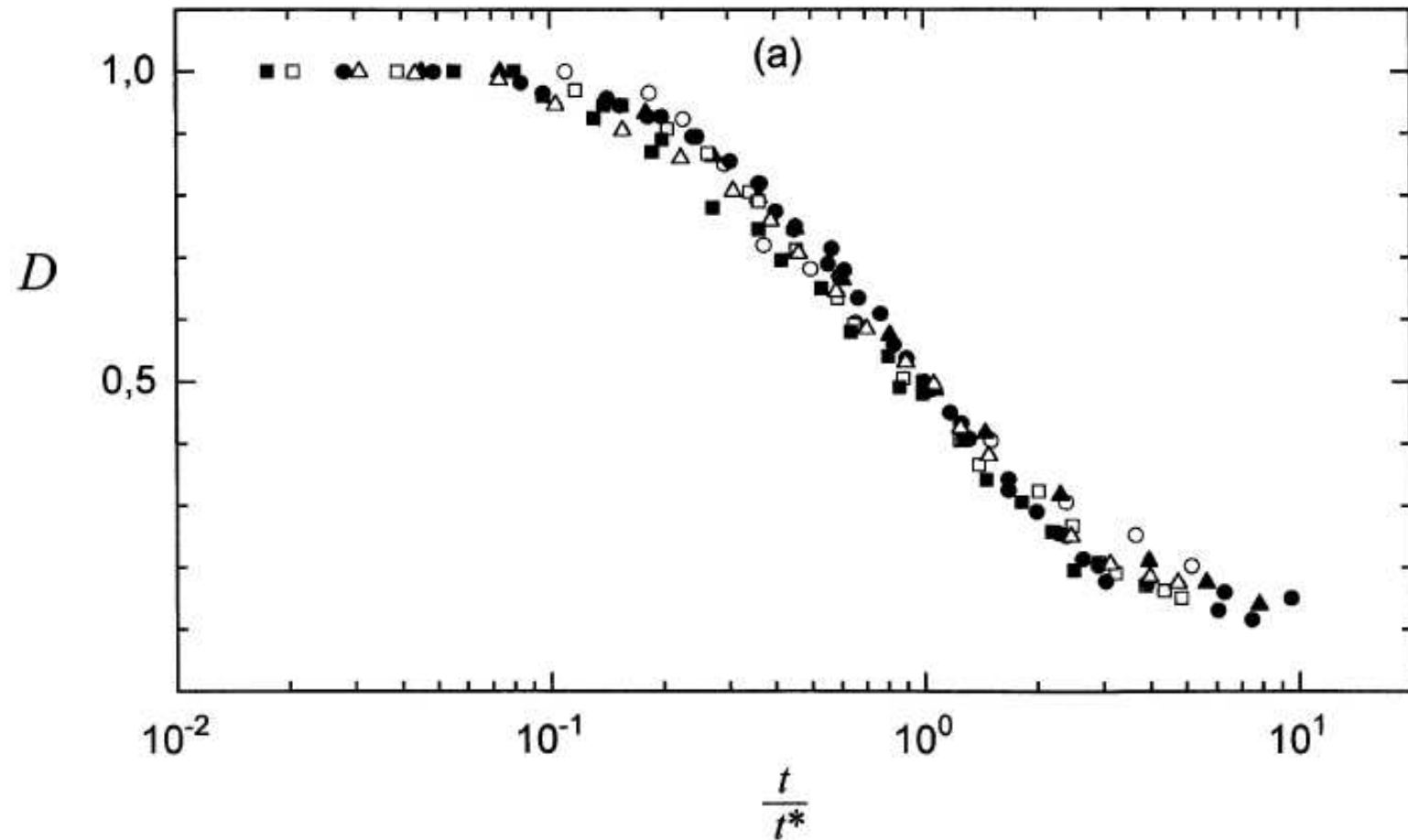
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\implies Some polymers survive breakage for large $\bar{\lambda} \tau$ if the fluctuations in $\bar{\lambda}$ (given by Δ) are also large.

Of course, this treatment is for one correlation time of the turbulence. In reality a given polymer is exposed to many different random strains.

Survival prob. decays roughly as $(\rho_c/\rho_m)^{2\zeta N}$, where $N = t/t_{\text{corr}}$ with t_{corr} the correlation time.

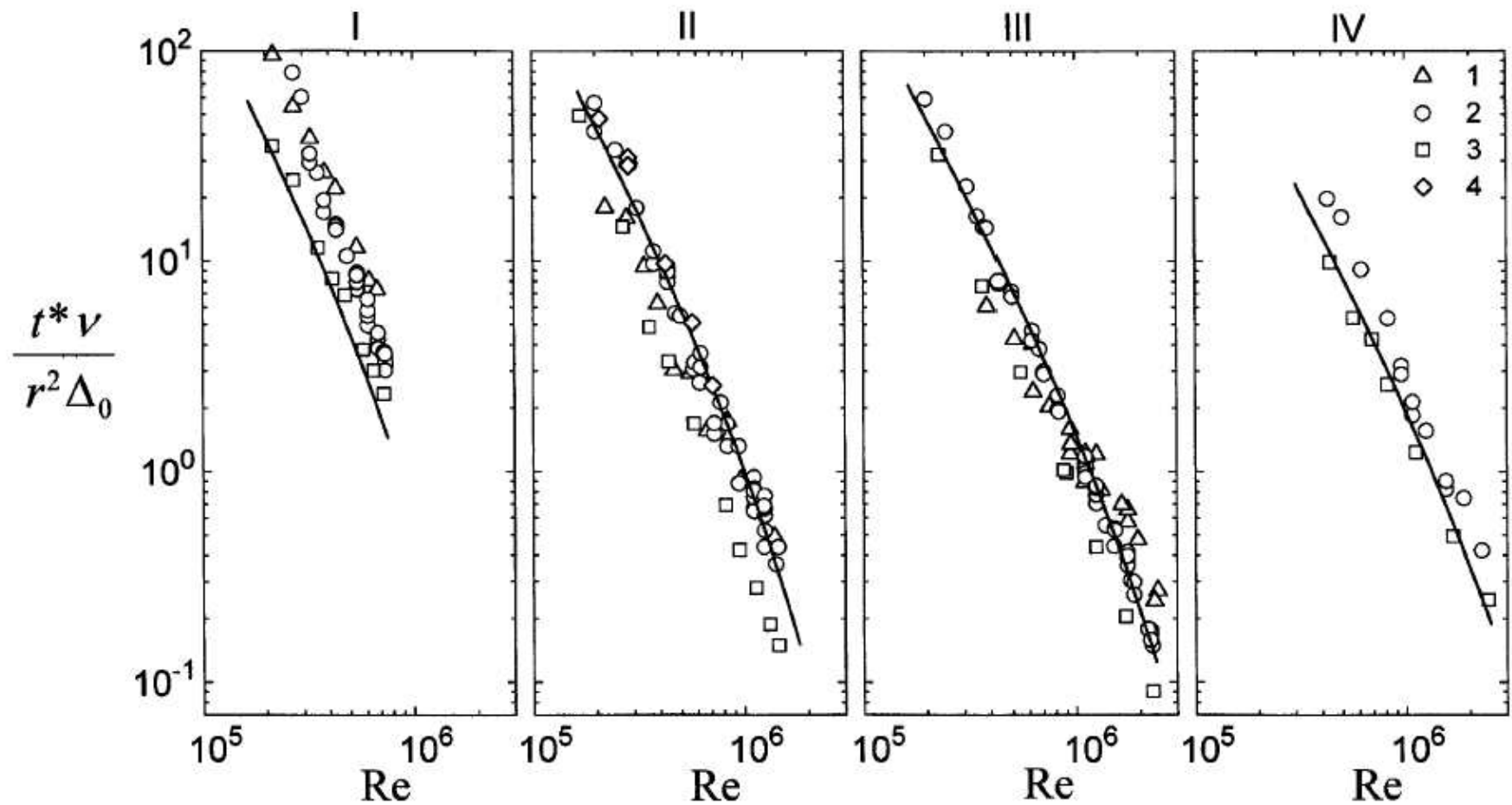
Degradation Experiments



From [Kalashnikov](#), *JNNFM* (2002)

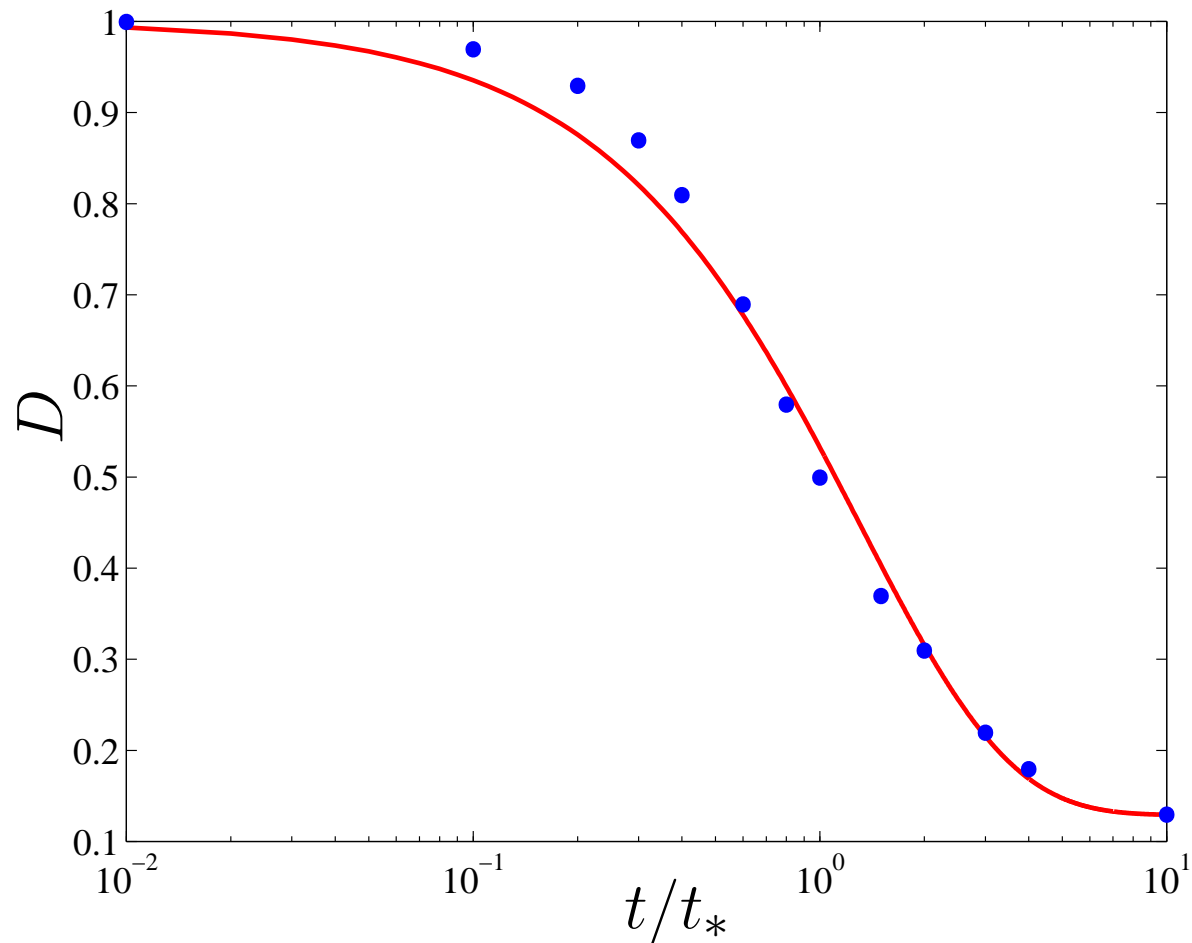
Half-degradation time

Reynolds number dependence:



Simple Fit to Exponential Decay

Early part not so important; also residual effect:



Predicts $t_* \sim t_{\text{corr}} \Delta / \text{Re}$

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- Statistics of curvature.