

# A Lagrangian approach to the kinematic dynamo

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with Allen Boozer

## Magnetic Field Evolution

The evolution of a magnetic field in resistive MHD is governed by the induction equation,

$$\frac{\partial \mathbf{B}}{\partial t} = \nabla \times (\mathbf{v} \times \mathbf{B}) + \frac{\eta}{\mu_0} \nabla^2 \mathbf{B}$$

where the **Eulerian** velocity field  $\mathbf{v}(\mathbf{x}, t)$  is some **prescribed** time-dependent flow.  $\mathbf{B}$  is the magnetic field,  $\eta$  is the resistivity, and  $\mu_0$  is the permeability of free space.

In a chaotic flow, fluid elements are **stretched** exponentially. The magnetic field grows due to the stretching, and the diffusion is also increased by this process. This enhancement is known as **chaotic mixing**.

## The Kinematic Dynamo

The **kinematic dynamo** involves the induction equation with the assumption that the magnetic field does not react back on the flow (the Lorentz force is neglected). This is justified when the field is small.

The **fast** kinematic dynamo problem can be formulated as follows:

Starting from a small seed magnetic field, what properties of  $\mathbf{v}$  are needed to obtain exponential growth of  $\mathbf{B}$ , such that the growth rate remains nonzero as  $\eta \rightarrow 0$ ?

This is relevant to astrophysical plasmas, where  $\eta/\mu_0 vL$  is so small that a growth rate going to 0 as  $\eta \rightarrow 0$  would be too small to account for observed variations of magnetic fields in, say, the sun.

## Lagrangian Coordinates

The trajectory of a fluid element in Eulerian coordinates  $\mathbf{x}$  satisfies

$$\frac{d\mathbf{x}}{dt}(\boldsymbol{\xi}, t) = \mathbf{v}(\mathbf{x}(\boldsymbol{\xi}, t), t)$$

where  $\boldsymbol{\xi}$  are **Lagrangian coordinates** that label fluid elements. The usual choice is to take as initial condition  $\mathbf{x}(\boldsymbol{\xi}, t = 0) = \boldsymbol{\xi}$ , which says that fluid elements are labeled by their initial position.

$\mathbf{x} = \mathbf{x}(\boldsymbol{\xi}, t)$  is thus the **transformation** from Lagrangian ( $\boldsymbol{\xi}$ ) to Eulerian ( $\mathbf{x}$ ) coordinates.

For a **chaotic flow**, this transformation gets horrendously complicated as time evolves.

## The Metric Tensor

The **Jacobian matrix** of the transformation  $\mathbf{x}(\boldsymbol{\xi}, t)$  is

$$M^i{}_q := \frac{\partial x^i}{\partial \xi^q}$$

For simplicity, we restrict ourselves to incompressible flows,  $\nabla \cdot \mathbf{v} = 0$ , so that  $\det M = 1$ . The Jacobian matrix is a precise record of how a fluid element is **rotated** and **stretched** by  $\mathbf{v}$ . We are interested in the stretching, not the rotation, so we construct the **metric tensor**

$$g_{pq} := \sum_{i=1}^3 M^i{}_p M^i{}_q$$

which contains only the information on the stretching of fluid elements.

## A Nonchaotic Example

As a simple demonstration, let us take the velocity field  $\mathbf{v}(x_1, x_2) = (0, f(x_1))$ , a shear flow in a channel. The Lagrangian trajectories are

$$\begin{aligned}x_1 &= \xi_1 \\x_2 &= \xi_2 + t f(\xi_1)\end{aligned}$$

The metric tensor is then

$$g_{pq} = \sum_{\ell} \frac{\partial x^{\ell}}{\partial \xi^p} \frac{\partial x^{\ell}}{\partial \xi^q} = \begin{pmatrix} 1 + t^2 f'(\xi_1)^2 & t f'(\xi_1) \\ t f'(\xi_1) & 1 \end{pmatrix}$$

The eigenvalues and eigenvectors of  $g$  are then easily derived. The eigenvectors converge **algebraically** to their time-asymptotic value.

## Stretching and Contracting Directions

The metric is a symmetric, positive-definite matrix, so it can be diagonalized with orthogonal eigenvectors  $\{\hat{\mathbf{u}}, \hat{\mathbf{m}}, \hat{\mathbf{s}}\}$  and corresponding real, positive eigenvalues  $\{\Lambda_u, \Lambda_m, \Lambda_s\}$ ,

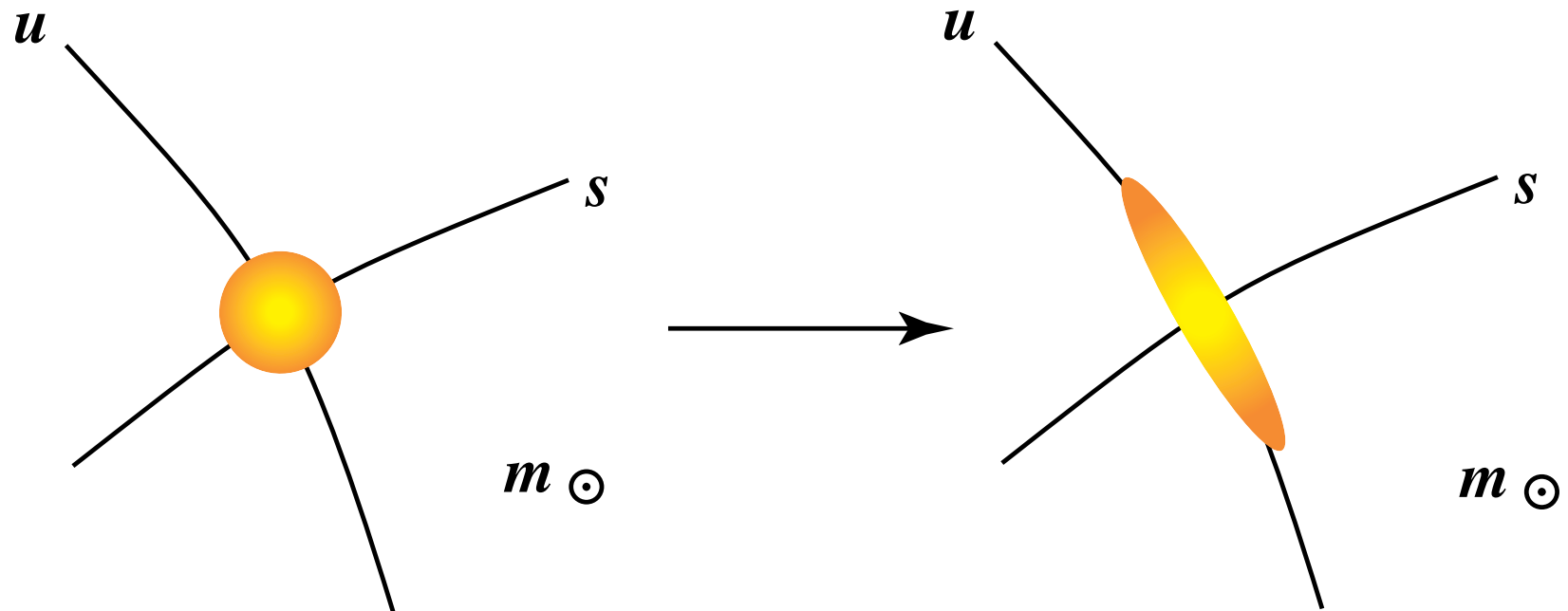
$$g_{pq} = \Lambda_u \hat{u}_p \hat{u}_q + \Lambda_m \hat{m}_p \hat{m}_q + \Lambda_s \hat{s}_p \hat{s}_q$$

The label  $u$  indicates an **unstable** direction: after some time  $\Lambda_u \gg 1$ , growing exponentially for long times. The label  $s$  indicates a **stable** direction: after some time  $\Lambda_s \ll 1$ , shrinking exponentially for long times. The **intermediate** direction, denoted by  $m$ , does not grow or shrink exponentially.

The  $\Lambda_\mu$  are related to the **finite-time Lyapunov exponents** by  $\lambda_\mu = \log \Lambda_\mu / 2t$ .

The incompressibility of  $\mathbf{v}$  implies that  $\Lambda_u \Lambda_m \Lambda_s = 1$ .

The eigenvalues and eigenvectors describe the **deformation** of a fluid element in a comoving frame:



The  $\hat{u}$  and  $\hat{s}$  directions can be integrated to yield the **unstable and stable manifolds**.

## ABC Flow

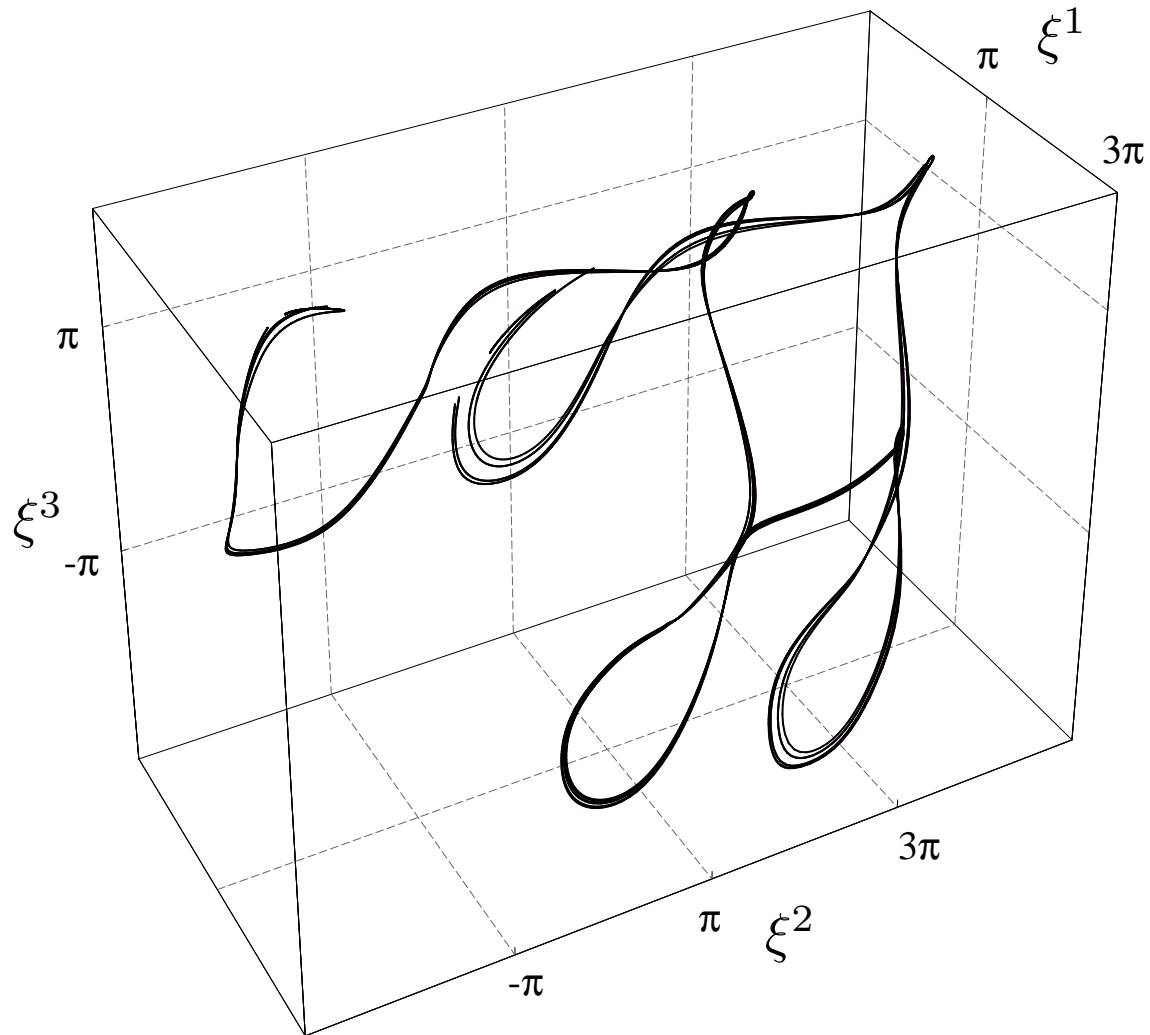
To exhibit the convergence of these quantities, we use the well-known **ABC** flow,

$$\mathbf{v}(\mathbf{x}) = A (0, \sin x_1, \cos x_1) + B (\cos x_2, 0, \sin x_2) + C (\sin x_3, \cos x_3, 0)$$

a sum of three **Beltrami waves**, which satisfy  $\nabla \times \mathbf{v} \propto \mathbf{v}$ . It is time-independent and incompressible ( $|g| = 1$ ).

We shall be using the parameter values  $A = 5, B = C = 2$  in subsequent examples.

Here's a portion of the **stable manifold**  $s(\xi)$  for the ABC 522 flow:



## Induction Equation in Lagrangian Coordinates

With the help of the metric tensor, we can transform the magnetic induction equation to Lagrangian coordinates  $\boldsymbol{\xi}$ :

$$\left. \frac{\partial}{\partial t} \right|_{\boldsymbol{\xi}} b^r = \sum_{p,q=1}^3 \frac{\eta}{\mu_0} \frac{\partial}{\partial \xi^p} \left[ g^{pq} \frac{\partial}{\partial \xi^q} b^r \right]$$

where  $b^r(\boldsymbol{\xi}, t) := \sum_i (M^{-1})^r_i B^i$  is the magnetic field in the Lagrangian frame, and  $g^{pq}(\boldsymbol{\xi}, t) := (g^{-1})^{pq}$ . The above equation is a diffusion equation with an **anisotropic** and **inhomogeneous** diffusivity,  $\eta g^{pq}$ . By construction, the velocity  $\mathbf{v}$  has dropped out of the equation entirely.

When  $\eta = 0$ , the above is the well-known result that in ideal MHD the magnetic field is frozen into the fluid.

## Diffusion of Magnetic Field

The diffusion equation involves the **inverse** metric  $g^{pq}$ , given in diagonal form by

$$g^{pq} = \Lambda_u^{-1} \hat{u}^p \hat{u}^q + \Lambda_m^{-1} \hat{m}^p \hat{m}^q + \Lambda_s^{-1} \hat{s}^p \hat{s}^q$$

The diffusion coefficients along the  $\hat{\mathbf{u}}$ ,  $\hat{\mathbf{m}}$ , and  $\hat{\mathbf{s}}$  directions are

$$D^{uu} = \sum_{p,q} \hat{u}_p (Dg^{pq}) \hat{u}_q = D\Lambda_u^{-1}$$

$$D^{mm} = \sum_{p,q} \hat{m}_p (Dg^{pq}) \hat{m}_q = D\Lambda_m^{-1}$$

$$D^{ss} = \sum_{p,q} \hat{s}_p (Dg^{pq}) \hat{s}_q = D\Lambda_s^{-1}$$

For a chaotic flow,  $D^{uu}$  goes to zero exponentially quickly,  $D^{mm}$  behaves algebraically, while  $D^{ss}$  **grows exponentially**.

Hence, **essentially all the diffusion occurs along the s-line**.

## The Dissipation Scale

Having a better feel for the geometry of the chaotic processes, such as the diffusion, we now attack a crucial question: **what is the scale of the smallest magnetic structures in the flow that will dissipate away?**

We define the length

$$\ell_B^2 = \frac{B^2}{\mu_0^2 j^2}$$

This definition is relevant because  $\eta j^2$  is the **Ohmic heating** term that dissipates magnetic energy. In a sense, the smaller  $\ell_B$ , the **less efficient** the dynamo is, because it will lose more of its energy to dissipation.

Brummell, Cattaneo, and Tobias (1998) define an averaged version of  $\ell_B$ , which is sensible in the Eulerian picture.

We stick with a local definition to take advantage of our Lagrangian formalism. We keep the vanishing resistivity and evaluate  $\ell_B$ .

Boozer (1992) proceeded as follows: the norm of the magnetic field is

$$B^2 = \sum_{p,q} b^p(\boldsymbol{\xi}) g_{pq}(\boldsymbol{\xi}, t) b^q(\boldsymbol{\xi})$$

The time dependence is contained [entirely inside the metric tensor](#). After a time, the stretching direction will dominate, so that

$$B^2 \propto \Lambda_u(\boldsymbol{\xi}, t) (b_u)^2,$$

where  $b_u := \mathbf{b} \cdot \hat{\mathbf{u}}$ .  $B^2$  grows exponentially, reflecting the chaotic amplification of the  $\mathbf{B}$  field by the flow.

The denominator of  $\ell_B$ ,  $(\mu_0 j)^2$ , can also be evaluated: its dominant part is

$$(\mu_0 j)^2 \propto \Lambda_u^3 (b_u \hat{\mathbf{u}} \cdot \nabla_0 \times \hat{\mathbf{u}})^2$$

where  $\nabla_0$  denotes a gradient with respect to the Lagrangian coordinates  $\xi$ .

Hence, the dissipation scale is

$$\ell_B^2 = \Lambda_u^{-2} (\hat{\mathbf{u}} \cdot \nabla_0 \times \hat{\mathbf{u}})^{-2}$$

This indicates that  $\ell_B$  decreases extremely rapidly. The efficiency of our dynamo is **getting exponentially worse** with time.

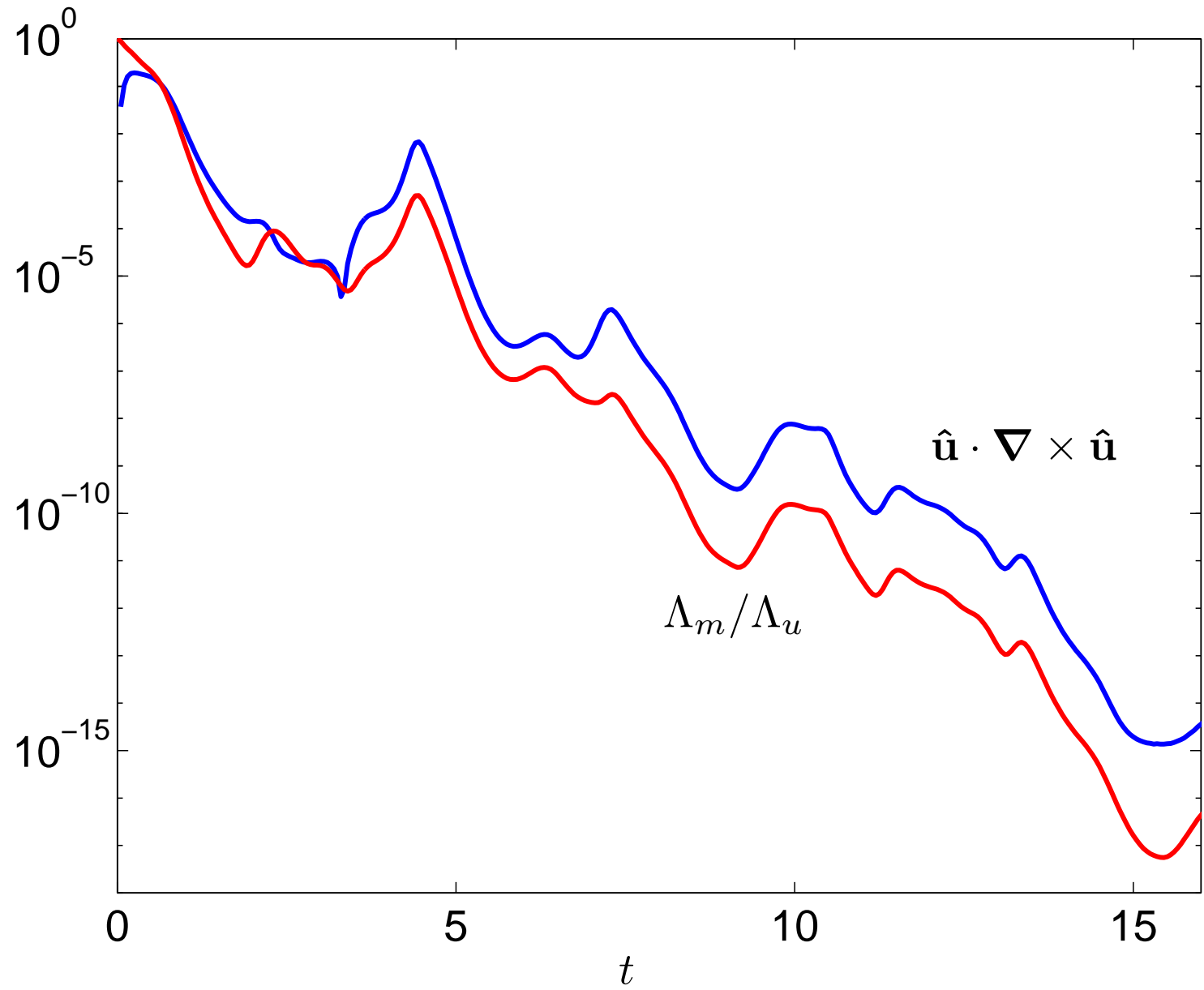
## ... But there are constraints!

We have applied [Thiffeault and Boozer, to appear in Chaos] the techniques of differential geometry to study the coordinate transformation  $\mathbf{x}(\boldsymbol{\xi}, t)$ . The metric  $g$  defines a **curvature tensor**, which is an invariant under coordinate transformations. Since the curvature **vanishes** in the Euclidean frame, it must do so in all coordinate systems, including the Lagrangian frame.

Enforcing this curvature condition, we derive **constraints** on the asymptotic form of  $\{\hat{\mathbf{u}}, \hat{\mathbf{m}}, \hat{\mathbf{s}}\}$  and  $\{\Lambda_u, \Lambda_m, \Lambda_s\}$ . The constraint we shall use here is

$$\hat{\mathbf{u}} \cdot \nabla_0 \times \hat{\mathbf{u}} \sim \Lambda_u^{-1} \Lambda_m \longrightarrow 0$$

The unstable direction line converges to an “helicity-free” configuration.



## Revised Dissipation Scale

Using the constraint in the current, we find that our earlier dominant term is now

$$(\mu_0 j)^2 \propto \Lambda_u \Lambda_m^{-2}$$

indicating that the current now grows at the same rate as  $B^2$ , since  $\Lambda_m \sim 1$  in an incompressible flow.

The problem is that the above  $j^2$  is essentially  $j_{\parallel} = \mathbf{j} \cdot \hat{\mathbf{u}}$ , which because of the constraint is no longer the dominant part of  $j^2$ .

The constraint shows that the dominant component of the current is not **parallel** to  $\mathbf{B}$  but **perpendicular** to it.

The form of the perpendicular part of  $\mathbf{j}$  is more complicated. After making use of another geometrical constraint, it can be shown that  $j^2$  grows faster than  $B^2$ , **unless** the initial condition  $\mathbf{b}$  is such that

$$\hat{\mathbf{s}} \cdot \nabla_0 \log \left( \tilde{\Lambda}_u^{-1/2}(\boldsymbol{\xi}) b_u(\boldsymbol{\xi}) \right) = 0$$

where  $\tilde{\Lambda}_u := e^{-2\lambda_u^\infty t} \Lambda_u$  is the trajectory separation with its asymptotic time behavior divided out (given by the **Lyapunov exponent**  $\lambda_u^\infty$ ). The function

$$\hat{\mathbf{s}} \cdot \nabla_0 \log \left( \tilde{\Lambda}_u^{-1/2} \right)$$

can be shown to converge rapidly to a constant value.

[Thiffeault, **in preparation**]

## Summary

- Using **Lagrangian coordinates**, we have revised earlier estimates of the behavior of the **energy dissipation scale**.
- We assumed **ideal evolution** and found the dominant terms in  $B^2$  and  $j^2$ , based on the assumption that the flow is **chaotic**.
- We adjusted the growth rate for new **geometrical constraints**.
- The constraint  $\hat{\mathbf{u}} \cdot \nabla_0 \times \hat{\mathbf{u}} \rightarrow 0$  tells us that the main contribution to the Ohmic heating comes from the current **perpendicular** to  $\mathbf{B}$ .
- For a special initial condition, we find a dissipation scale that **does not decrease exponentially**. This might be closely related to the **eigenfunction** of the induction operator that would not diffuse away when resistivity is present.