Chaotic Mixing in a Torus Map

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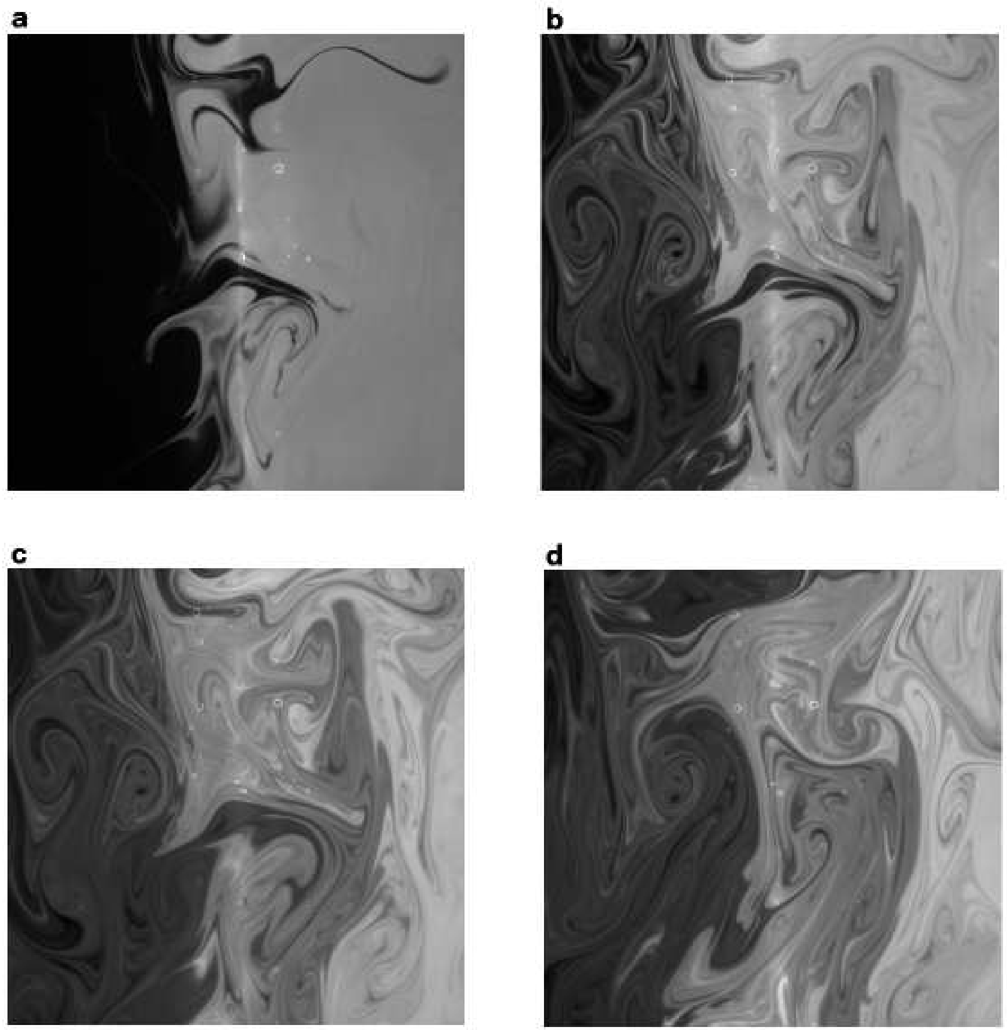
Experiment of Rothstein et al. (1999)

Regular array of magnets

[Rothstein, Henry, and Gollub, Nature **401**, 770 (1999)]

Persistent Pattern

Disordered array ($i = 2, 20, 50, 50.5$)



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- Applies if initial scale large, or for inverse cascade.
- [Pierrehumbert, Chaos Sol. Frac. (1994)] Strange eigenmode
  [Fereday et al., Wonhas and Vassilicos, PRE (2002)] Baker's map
  [Sukhatme and Pierrehumbert, PRE (2002)] Unified description

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- Today: Focus on Global theory.
- Map allows analytical results (enough said!).

The Map

We consider a diffeomorphism of the 2-torus $\mathbb{T}^2 = [0,1]^2$,

$$\mathcal{M}(\boldsymbol{x}) = \mathbb{M} \cdot \boldsymbol{x} + \phi(\boldsymbol{x}),$$

where

$$\mathbb{M} = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}; \qquad \phi(\boldsymbol{x}) = \frac{K}{2\pi} \begin{pmatrix} \sin 2\pi x_1 \\ \sin 2\pi x_1 \end{pmatrix};$$

$\mathbb{M} \cdot \boldsymbol{x}$ is the Arnold cat map.

The map $\mathcal{M}$ is area-preserving and chaotic.

For $K = 0$ the stretching of fluid elements is homogeneous in space.
For small K the system is still uniformly hyperbolic.

Advection and Diffusion

Iterate the map and apply the heat operator to a scalar field (which we call temperature for concreteness) distribution $\theta^{(i-1)}(\boldsymbol{x})$,

$$\theta^{(i)}(\boldsymbol{x}) = \mathcal{H}_\epsilon\,\theta^{(i-1)}(\mathcal{M}^{-1}(\boldsymbol{x}))$$

where ϵ is the diffusivity, with the heat operator $\mathcal{H}_\epsilon$ and kernel h_ϵ

$$\mathcal{H}_\epsilon\theta(\boldsymbol{x}) := \int_{\mathbb{T}^2} h_\epsilon(\boldsymbol{x}-\boldsymbol{y})\theta(\boldsymbol{y})\,\mathrm{d}\boldsymbol{y};$$

$$h_\epsilon(\boldsymbol{x}) = \sum_{\boldsymbol{k}} \exp(2\pi\mathrm{i}\boldsymbol{k}\cdot\boldsymbol{x} - \boldsymbol{k}^2\epsilon).$$

In other words: advect instantaneously and then diffuse for one unit of time.

Transfer Matrix

Fourier expand $\theta^{(i)}(\boldsymbol{x})$,

$$\theta^{(i)}(\boldsymbol{x}) = \sum_{\boldsymbol{k}} \hat{\theta}^{(i)}_{\boldsymbol{k}} \mathrm{e}^{2\pi\mathrm{i}\boldsymbol{k}\cdot\boldsymbol{x}} .$$

The effect of advection and diffusion becomes

$$\hat{\theta}^{(i)}(\boldsymbol{x}) = \sum_{\boldsymbol{q}} \mathbb{T}_{\boldsymbol{k}\boldsymbol{q}}\, \hat{\theta}^{(i-1)}_{\boldsymbol{q}},$$

with the transfer matrix,

$$\begin{aligned}\mathbb{T}_{\boldsymbol{k}\boldsymbol{q}} &:= \int_{\mathbb{T}^2} \exp\left(2\pi\mathrm{i}\left(\boldsymbol{q}\cdot\boldsymbol{x} - \boldsymbol{k}\cdot\mathcal{M}(\boldsymbol{x})\right) - \epsilon\,\boldsymbol{q}^2\right)\,\mathrm{d}\boldsymbol{x}, \\ &= \mathrm{e}^{-\epsilon\,\boldsymbol{q}^2}\,\delta_{0,Q_2}\,\mathrm{i}^{Q_1}\,J_{Q_1}\left(\left(k_1+k_2\right)K\right), \qquad \boldsymbol{Q} := \boldsymbol{k}\cdot\mathbb{M} - \boldsymbol{q},\end{aligned}$$

where the J_Q are the Bessel functions of the first kind.

Variance: A measure of mixing

In the absence of diffusion ($\epsilon = 0$), the variance $\sigma^{(i)}$

$$\sigma^{(i)} := \int_{\mathbb{T}^2} \left|\theta^{(i)}(\boldsymbol{x})\right|^2 \mathrm{d}\boldsymbol{x} = \sum_{\boldsymbol{k}} \sigma^{(i)}_{\boldsymbol{k}}, \qquad \sigma^{(i)}_{\boldsymbol{k}} := \left|\hat{\theta}^{(i)}_{\boldsymbol{k}}\right|^2$$

is preserved. (We assume the spatial mean of θ is zero.)

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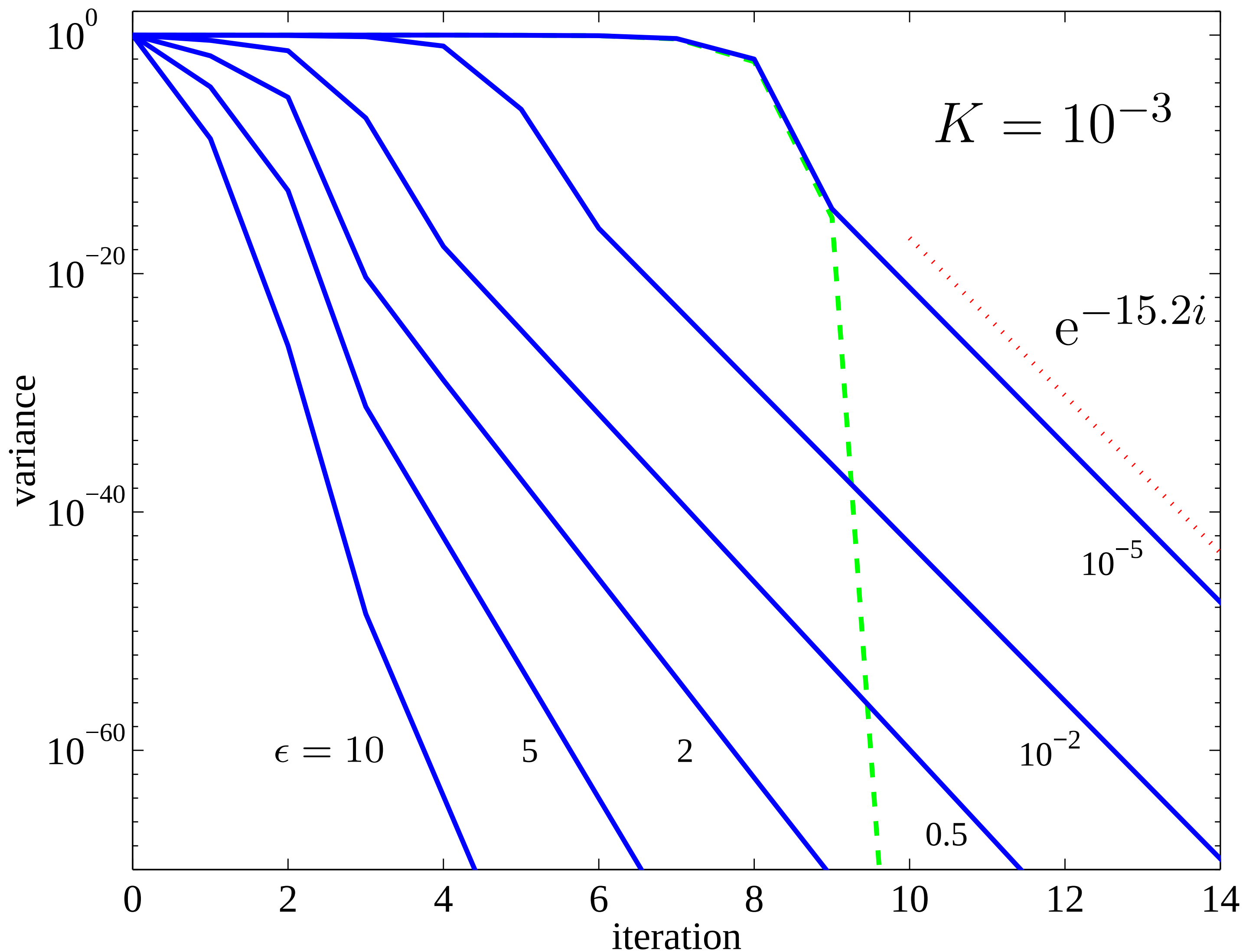
- $\theta^{(i)}$ settles into an eigenfunction of the A–D operator that sets the exponential decay rate.

Decay of Variance



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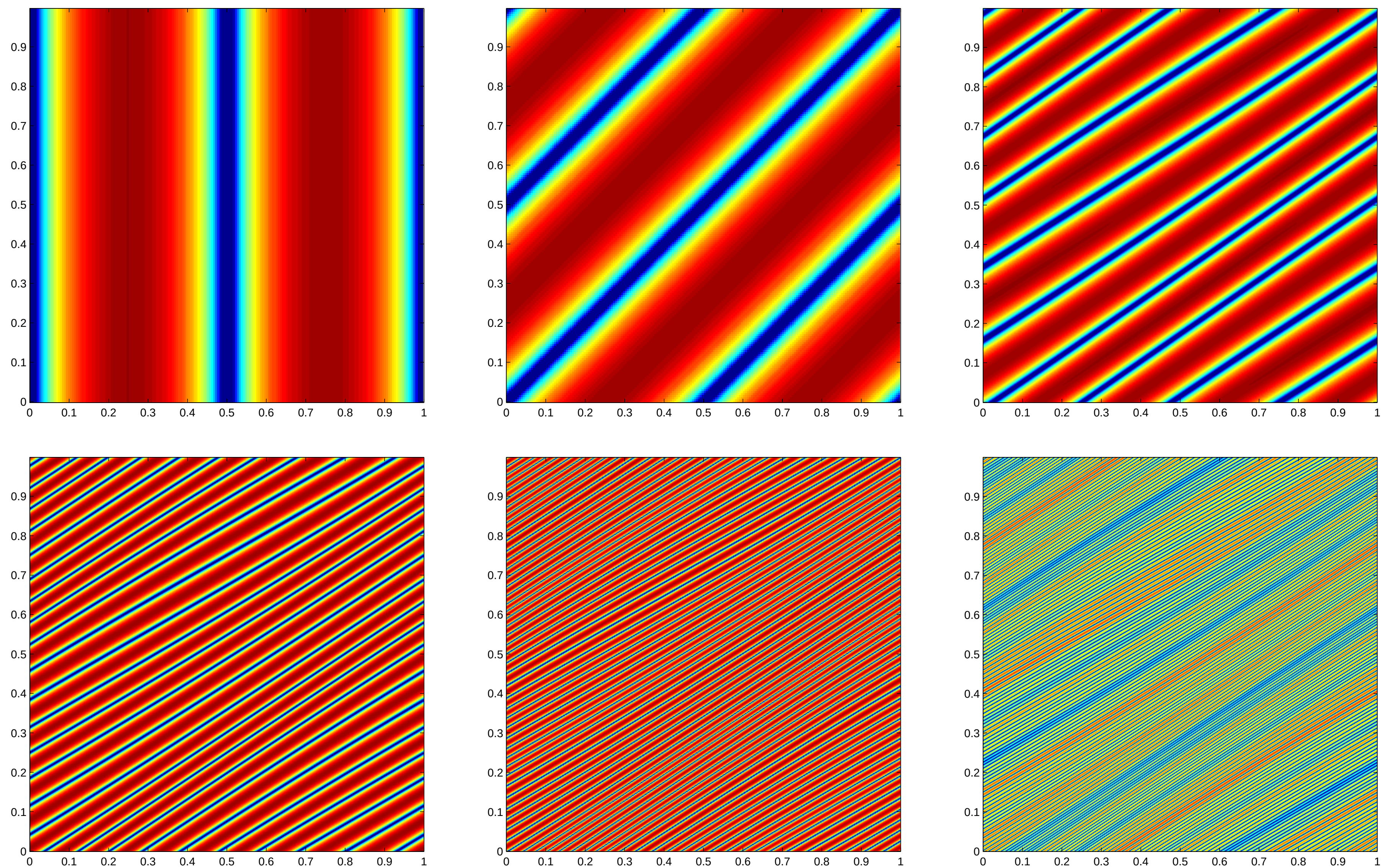
- This is the well-known “filamentation” effect in chaotic flows: the stretching and folding action of the flow causes rapid variation of the temperature across the folds.
- Can no longer neglect diffusion after a number of iterations

$$i_1 \simeq 1 + (\log \epsilon^{-1} / \log \Lambda^2) \simeq 6 \quad \text{for } \epsilon = 10^{-5},$$

where $\Lambda = (3 + \sqrt{5})/2$ is the largest eigenvalue of $\mathbb{M}^{-1}$.

Variance: 5 iterations for $K = 0.3$ and $\epsilon = 10^{-3}$



Superexponential Phase

For small K and $\boldsymbol{k}$, we have $J_0\left((k_1+k_2)K\right) \gg J_1\left((k_1+k_2)K\right)$. Set $K=0$ and retain only the $Q_1=0$ term in the transfer matrix,

$$\mathbb{T}_{\boldsymbol{k}\boldsymbol{q}} = \mathrm{e}^{-\epsilon\,\boldsymbol{q}^2}\,\delta_{\boldsymbol{k},\,\boldsymbol{q}\cdot\mathbb{M}^{-1}} + \mathcal{O}\Big((k_1+k_2)^2K^2\Big)\,;$$

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If initially the variance is concentrated in a single wavenumber $\boldsymbol{q}_0$, then after one iteration it will all be in $\boldsymbol{q}_0\cdot\mathbb{M}^{-1}$, after two in $\boldsymbol{q}_0\cdot\mathbb{M}^{-2}$, etc.

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The length of $\boldsymbol{q}$ is multiplied by the eigenvalue Λ at each iteration.

But each at each step the variance is multiplied by the diffusive decay factor $\exp(-\epsilon\,\boldsymbol{q}^2)$, with $\boldsymbol{q}$ getting exponentially larger.

The net decay is thus superexponential.

Exponential Phase

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- This direct cascade process dominates at first, but it is so efficient that eventually we must examine the effect of the the wave term (sin), which is felt through the higher-order Bessel functions in the transfer matrix.
- Can the wave term lead to the formation of an eigenfunction of the advection–diffusion operator, which would imply exponential decay?

An Eigenfunction?

Recall:

$$\mathbb{T}_{\boldsymbol{kq}} = \mathrm{e}^{-\epsilon\,\boldsymbol{q}^2}\,\delta_{0,Q_2}\,\mathrm{i}^{Q_1}\,J_{Q_1}\left(\left(k_1 + k_2\right)K\right), \qquad \boldsymbol{Q} := \boldsymbol{k}\cdot\mathbb{M} - \boldsymbol{q},$$

Consider a matrix element for which $Q_1 \neq 0$. This means that the initial ($\boldsymbol{q}$) and final ($\boldsymbol{k}$) wavenumbers connected by that matrix element can differ from $\boldsymbol{k}\cdot\mathbb{M} = \boldsymbol{q}$ by Q_1 in their first component.

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Is it possible for a wavenumber to be mapped back onto itself by such a coupling? Seek solutions to

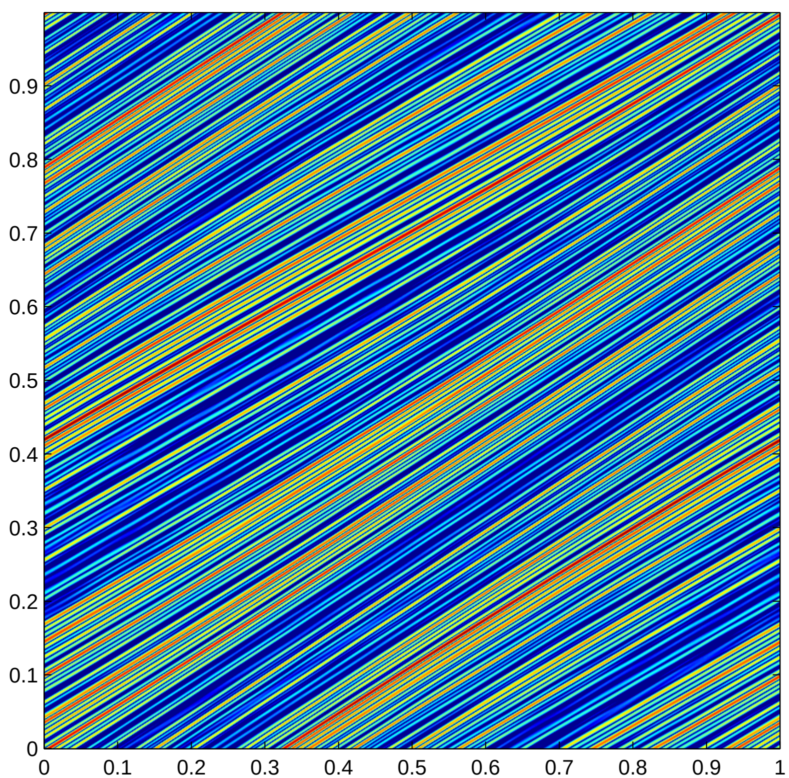
$$(q_1 \;\; q_2)\cdot\mathbb{M} = (q_1 + Q_1 \;\; q_2) \qquad \Longrightarrow \qquad (q_1 \;\; q_2) = (0 \;\; Q_1)\,.$$

The matrix element connecting the $(0 \;\; Q_1)$ mode to itself is

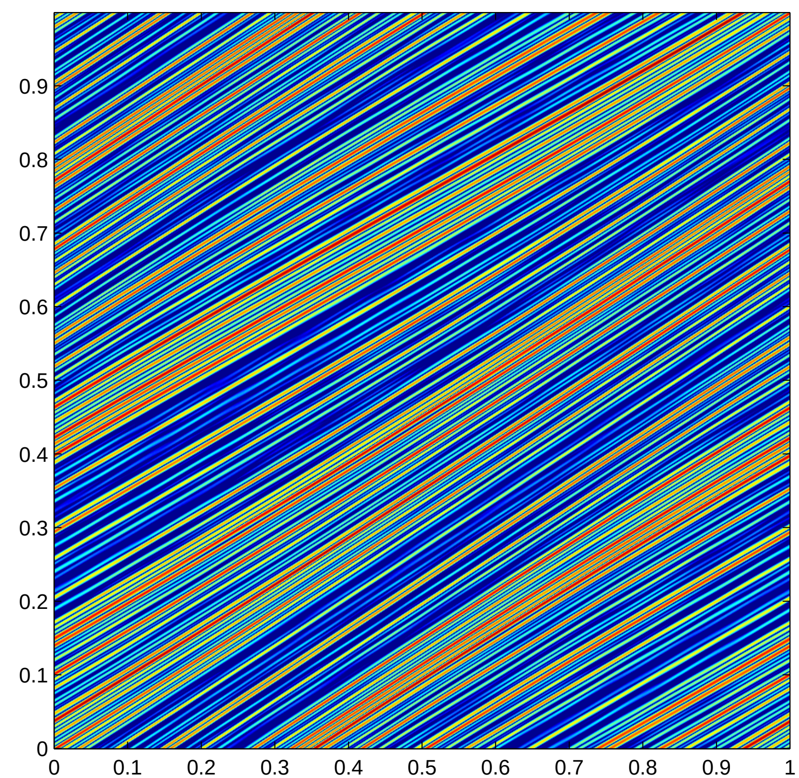
$$\mathbb{T}_{(0\;Q_1),(0\;Q_1)} = \mathrm{e}^{-\epsilon\,Q_1^2}\,\mathrm{i}^{Q_1}\,J_{Q_1}\left(Q_1\,K\right).$$

Eigenfunction for $K = 0.3$ and $\epsilon = 10^{-3}$

(Renormalised by decay rate)



$i = 25$



$i = 30$

Decay Rate

For small K, the dominant Bessel function is J_1, so the decay factor μ^2 for the variance is given by

$$\mu = \left|\mathbb{T}_{(0\ 1),(0\ 1)}\right| = \mathrm{e}^{-\epsilon}\, J_1\,(K) = \tfrac{1}{2}K + \mathcal{O}\big(\epsilon\, K, K^2\big)\,.$$

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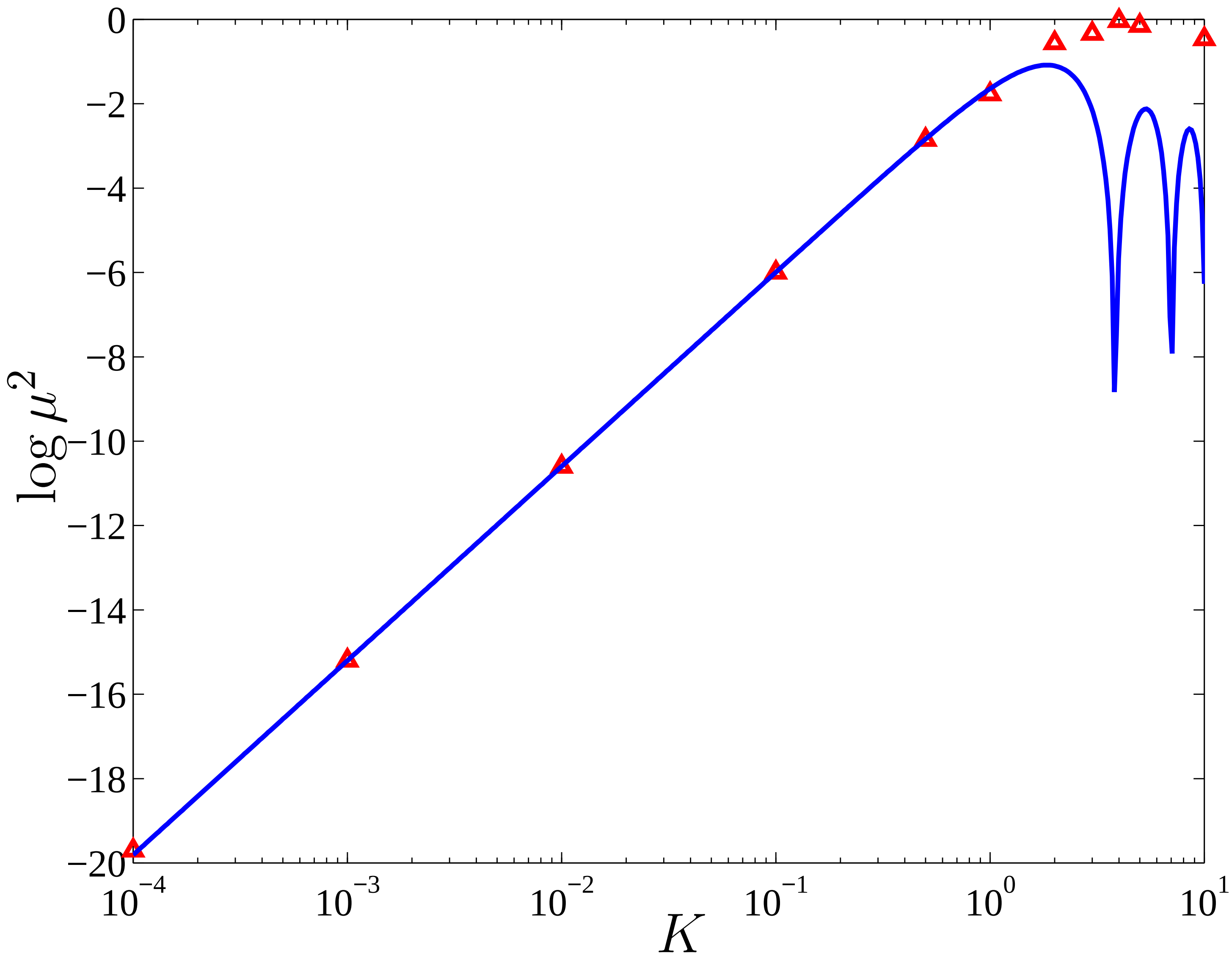
This is an analogous result to the baker's map [Fereday et al., Wonhas and Vassilicos, PRE (2002)], except that here the agreement with numerical results is good for K quite close to unity.

This is because in the baker's map the discontinuity generates many slowly-decaying harmonics at each step.

Decay Rate as $\epsilon \to 0$



Variance Spectrum of the Eigenfunction

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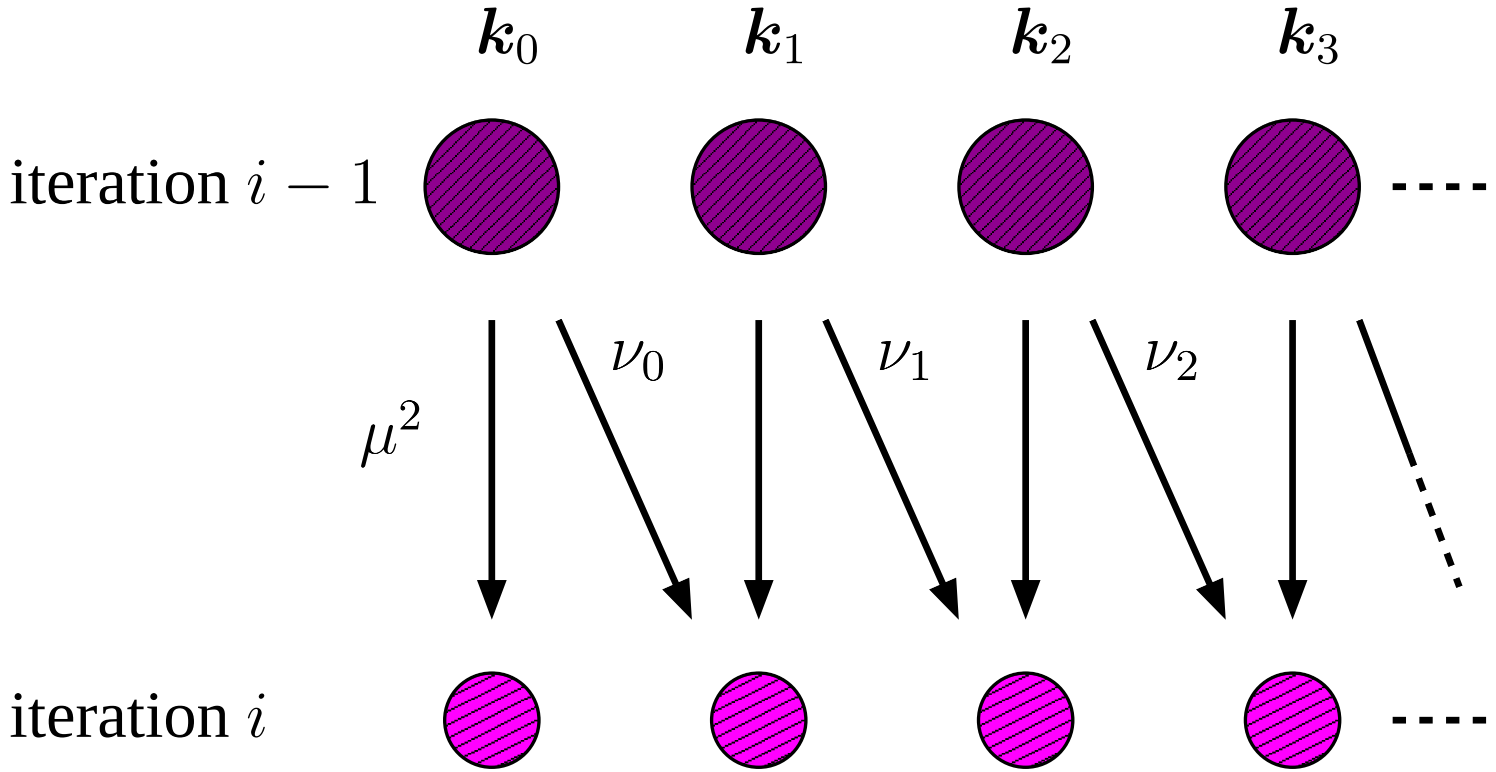
- But this dominant mode does not determine the structure of the eigenfunction.
- In fact, a very small amount of the total variance actually resides in that bottleneck mode: the variance is concentrated at small scales.

Eigenfunction: One Iteration

The wavenumbers are mapped back to themselves, with their variance decreased by a uniform factor $\mu^2 < 1$ (vertical arrows). But at the same time the modes are mapped to next one down the cascade following the diagonal arrows.



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If we denote by $\sigma_n^{(i)} := |\hat{\theta}_{\boldsymbol{k}_n}|^2$ the variance in mode $\boldsymbol{k}_n$ at the ith iteration, we have

$$\sigma_n^{(i)} = \mu^2\, \sigma_n^{(i-1)}, \qquad n = 0, 1, \dots,$$

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These two recurrences can be combined to give

$$\Sigma_n^{(i)} := \frac{\sigma_n^{(i)}}{\sigma_0^{(i)}} = \frac{\nu_{n-1}\, \nu_{n-2}\, \cdots \nu_0}{\mu^{2n}} = \mu^{-2n} \exp\left(-2\epsilon \sum_{m=0}^{n-1} \boldsymbol{k}_m^2\right),$$

where $\Sigma_n^{(i)}$ is the relative variance in the nth mode.

Eigenfunction and Cascade (cont'd)

The wavenumber is given by the exponential recursion,

$$\|\boldsymbol{k}_n\| \simeq \Lambda\,\|\boldsymbol{k}_{n-1}\| \quad \Longrightarrow \quad \|\boldsymbol{k}_n\| \simeq \Lambda^n\,\|\boldsymbol{k}_0\| = \Lambda^n\,.$$

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Solve for $n = \log\|\boldsymbol{k}_n\|\,/\,\log\Lambda$ and rewrite the relative variance as

$$\Sigma_n^{(i)} \simeq \|\boldsymbol{k}_n\|^{-2\log\mu/\log\Lambda}\exp\left(-2\epsilon\boldsymbol{k}_n^2/\Lambda^2\right),$$

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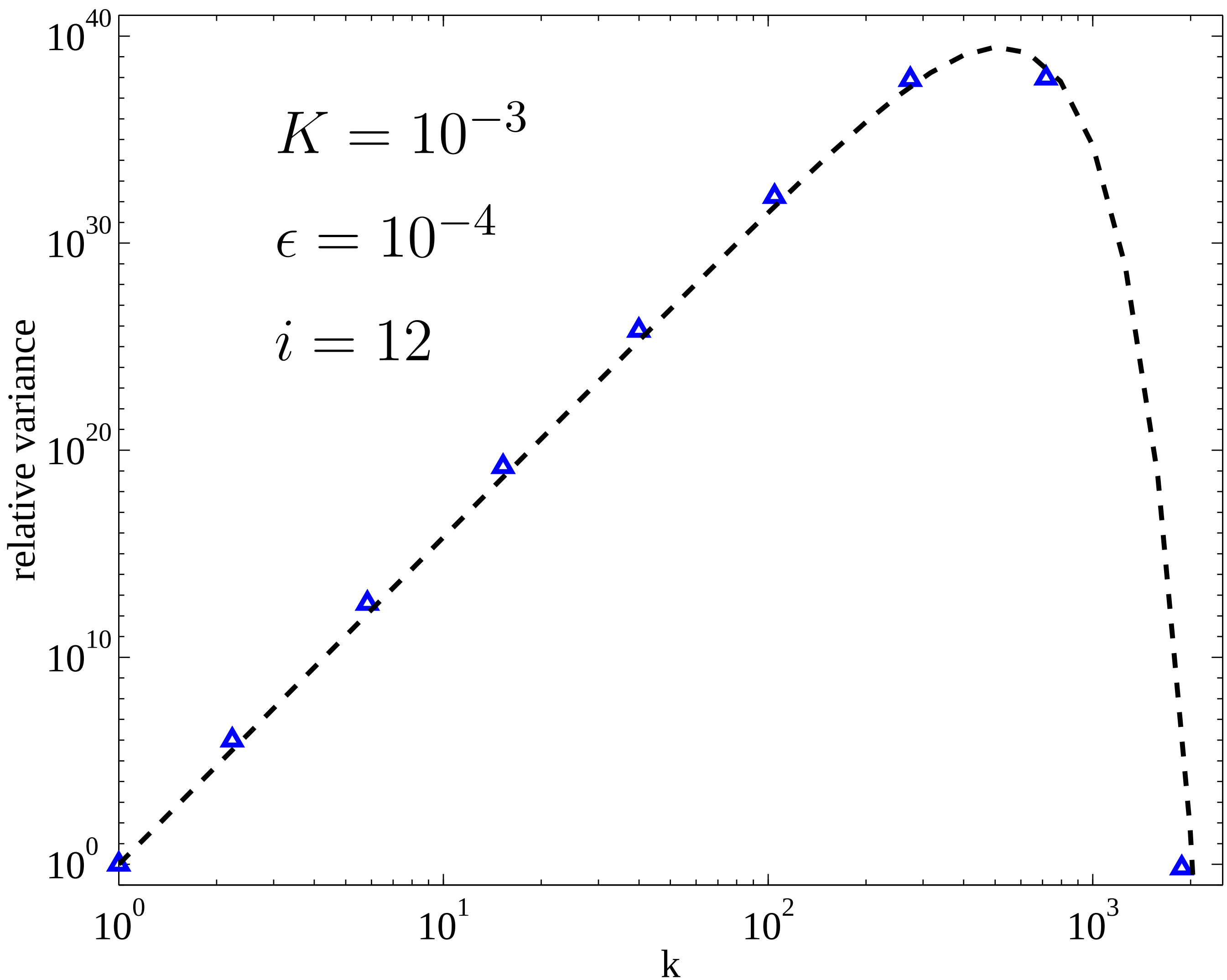
Does not (and should not) depend on the iteration number, i, and depends only on n through $\boldsymbol{k}_n$. Find

$$\boxed{\Sigma(k) = k^{2\zeta} \exp\left(-2\epsilon k^2/\Lambda^2\right) , \qquad \zeta := -\log\mu/\,\log\Lambda,}$$

the spectrum of relative variance.

Spectrum of Variance



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- Large K? Periodic orbits?