

Fluid Equations for Rarefied Gases

Jean-Luc Thiffeault

*Department of Applied Physics and Applied Mathematics
Columbia University*

<http://plasma.ap.columbia.edu/~jeanluc>

23 March 2001

with E. A. Spiegel

Fluid Equations

- Equations for **macroscopic variables**: ρ , \mathbf{u} , T .
- **Navier–Stokes** is the most well-known. Discovered empirically.
- **Hilbert** and **Chapman & Enskog** derived N–S as an expansion of the **Boltzmann equation** in the **Knudsen number** ϵ , the mean-free-path over macroscopic length scales.
- Domain of validity is limited: does poorly with rarefied gases ($\epsilon \sim 1$).
- Higher order expansions (**Burnett**) do not seem to do much better, and can actually do **worse**.

Asymptotic but not Convergent

Grad (1963) expressed it best:

“The apparent ineffectiveness of the Burnett equations...is evidently a consequence of the fact that the Chapman–Enskog expansion is **asymptotic** rather than **convergent**. **The series must always be truncated**. For rough answers,...one should use the Navier–Stokes equations (or give up the Chapman–Enskog approach entirely).”

Propose a modification of Chapman–Enskog in an attempt to extend its domain of validity.

[**BGK**: Chen, Rao, and Spiegel (2000); **Fokker–Planck**: Spiegel and Thiffeault (2001)]

The Boltzmann Equation

$$\mathcal{D}f = \frac{1}{\epsilon} \mathcal{C}[f]$$

Liouville operator:

$$\mathcal{D} \equiv \partial_t + v^i \partial_{x^i} + K^i \partial_{v^i}$$

Collision operator satisfies

$$\int \psi^\alpha \mathcal{C}[f] d^3v = 0, \quad \alpha = 1, \dots, 5,$$

where the ψ^α are the **collisional invariants**,

$$\psi^\alpha = m \left(1, \mathbf{v}, \frac{1}{2} v^2 \right)$$

There are **no other collisional invariants**.

Macroscopic Variables

Can define “slow variables,”

$$\rho \equiv \int m f d^3v, \quad \mathbf{u} \equiv \frac{1}{\rho} \int m \mathbf{v} f d^3v, \quad T \equiv \frac{m}{3R\rho} \int c^2 f d^3v,$$

where the peculiar velocity is

$$\mathbf{c}(\mathbf{x}, \mathbf{v}, t) \equiv \mathbf{v} - \mathbf{u}(\mathbf{x}, t).$$

There are no other slow variables.

Fluid Equations

From the collisional invariants, can derive fluid equations for the slow variables,

$$\partial_t \rho + \nabla \cdot (\rho \mathbf{u}) = 0, \quad (1)$$

$$\partial_t \mathbf{u} + \mathbf{u} \cdot \nabla \mathbf{u} = -\rho^{-1} \nabla \cdot \mathbb{P} + \mathbf{K}, \quad (2)$$

$$\frac{3}{2} \rho R (\partial_t T + \mathbf{u} \cdot \nabla T) = -\mathbb{P} : \nabla \mathbf{u} - \nabla \cdot \mathbf{Q}. \quad (3)$$

where

$$\mathbb{P} \equiv \int m \mathbf{c} \mathbf{c} f d^3 v, \quad \mathbf{Q} \equiv \int \frac{1}{2} m c^2 \mathbf{c} f d^3 v, \quad (4)$$

are the **pressure tensor** and the **heat flux**.

Everything is still **exact**. But what are \mathbb{P} and \mathbf{Q} ?

Mean-free-path Expansion

Hilbert and Chapman & Enskog suggested an expansion of f in the mean-free-path, as expressed by the Knudsen number, ϵ .

The equilibrium state is defined by $\mathcal{C}[f_0] = 0$, and is given by the Maxwellian distribution,

$$f_0(\mathbf{x}, \mathbf{v}, t) = \frac{n}{(2\pi RT)^{3/2}} \exp\left(-\frac{c^2}{2RT}\right).$$

At order ϵ^0 , we have $f_{(0)} = f_0$, and we recover the Euler equations, for which

$$\mathbb{P} = p \mathbb{I}, \quad \mathbf{Q} = 0,$$

where $p = \rho RT$ is the scalar pressure (ideal gas law).

Fokker–Planck Operator

Need to specify collision operator \mathcal{C} . Use [Fokker–Planck](#),

$$\mathcal{C}[f] \equiv \sum_{i,j} \frac{\partial}{\partial v^i} \left[-A^i f + \frac{1}{2} \frac{\partial}{\partial v^j} (B^{ij} f) \right]$$

[random walk](#) in velocity space.

Used in plasma physics, astrophysics...

Conserves collisional invariants ψ^α if

$$A^i = -\nu(v^i - u^i), \quad B^{ij} = 2\nu RT \delta^{ij}$$

Absorb the [collision frequency](#) ν in ϵ .

First Order

Write the first-order solution $f_{(1)}$ as a polynomial in the peculiar velocity $\mathbf{c} = \mathbf{v} - \mathbf{u}$, times a Maxwellian,

$$\begin{aligned} f_{(1)} &= \sum \left(\alpha_{ijk}^{(3)} c^i c^j c^k + \alpha_{ij}^{(2)} c^i c^j + \alpha_i^{(1)} c^i + \alpha^{(0)} \right) f_0 \\ &\equiv \tilde{f}_{(1)} f_0, \end{aligned}$$

then solve the differential equation

$$\mathcal{C}[f_{(1)}] = \mathcal{D}f_0$$

by equating powers of \mathbf{c} .

Pressure Tensor and Heat Flux

After solving for $f_{(1)}$ and evaluating the first-order pressure tensor and heat flux, find

$$\mathbb{P} = p \left[\mathbb{I} - \frac{1}{2} \epsilon \left(\nabla \mathbf{u} + \widetilde{\nabla \mathbf{u}} - \frac{2}{3} \nabla \cdot \mathbf{u} \mathbb{I} \right) - \frac{1}{2} \epsilon \left(\frac{D \ln T}{Dt} + \frac{2}{3} \nabla \cdot \mathbf{u} \right) \mathbb{I} \right],$$

$$\mathbf{Q} = -\frac{5}{2} p \epsilon \left[\frac{1}{3} R \nabla T + \left(\frac{D \mathbf{u}}{Dt} - \mathbf{K} \right) + R T \nabla \ln p \right],$$

where the [convective derivative](#) is

$$\frac{D}{Dt} \equiv \frac{\partial}{\partial t} + \mathbf{u} \cdot \nabla$$

How do we eliminate those convective derivatives?

The Chapman–Enskog Result

Chapman & Enskog use the Euler equations:

$$\begin{aligned}\mathbb{P} - p\mathbb{I} &= -\frac{1}{2} p \epsilon \overset{\circ}{\mathbb{E}}, \\ \mathbf{Q} &= -\frac{5}{6} p \epsilon R \nabla T,\end{aligned}$$

where the traceless rate-of-strain tensor is

$$\overset{\circ}{\mathbb{E}} \equiv \nabla \mathbf{u} + \widetilde{\nabla \mathbf{u}} - \frac{2}{3} \nabla \cdot \mathbf{u} \mathbb{I}$$

This gives the Navier–Stokes equations.

But why use Euler?

If instead we simply insert \mathbb{P} and \mathbf{Q} into the **full fluid equations**, we obtain

$$\partial_t \mathbf{u} + \mathbf{u} \cdot \nabla \mathbf{u} = -\frac{1}{\rho} \nabla \left(p - \frac{1}{2} \frac{\epsilon}{C_v} \frac{DS}{Dt} \right) - \frac{1}{2} \rho^{-1} \nabla \cdot (p \epsilon \overset{\circ}{\mathbb{E}}),$$

and a similarly modified equation for T . We have introduced the **specific entropy**,

$$S = C_v \ln \left(\frac{p}{\rho^{5/3}} \right).$$

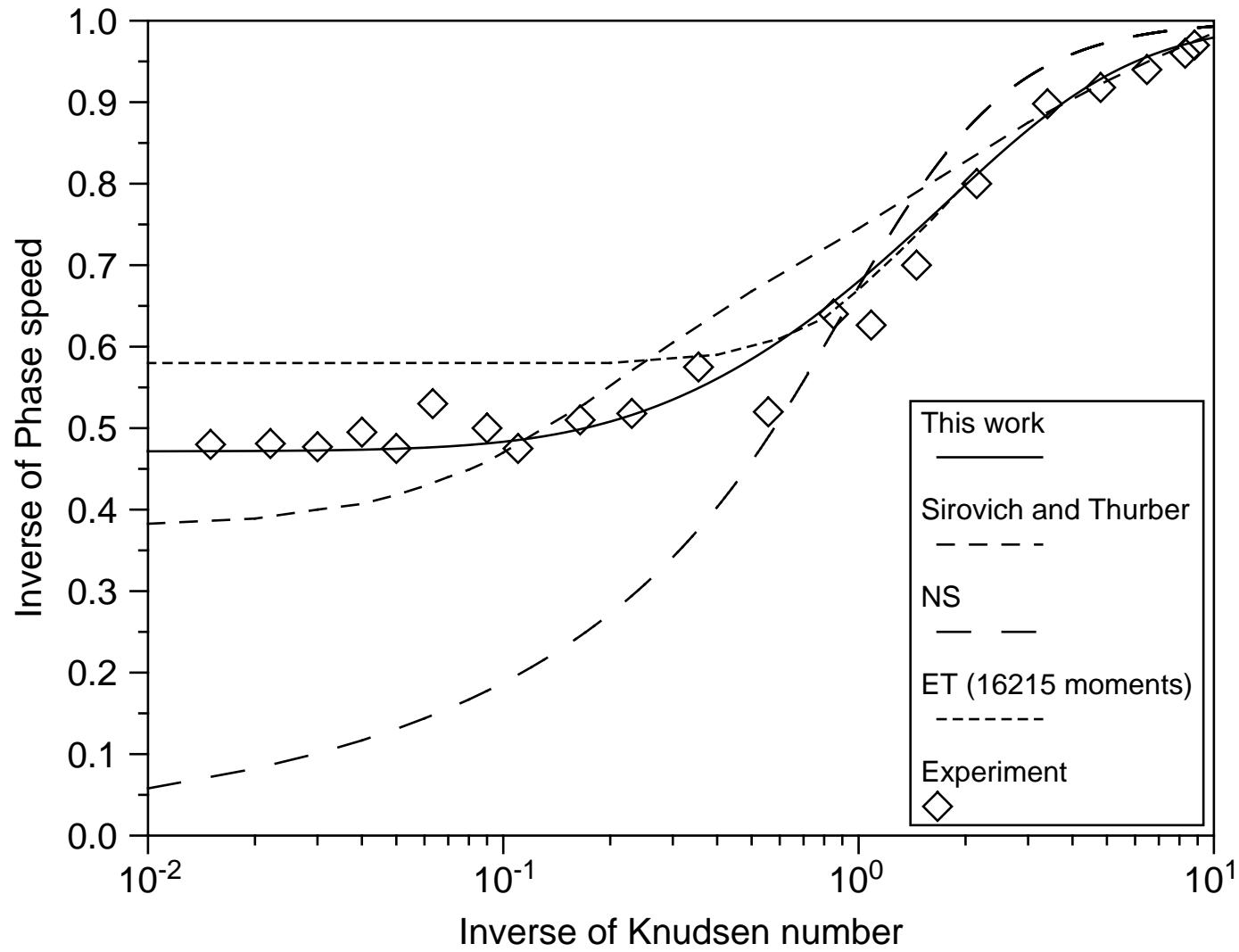
There is a **bulk viscosity** in the equation, given by the rate of entropy production.

Implicit differential equation containing terms of **all orders** in ϵ .

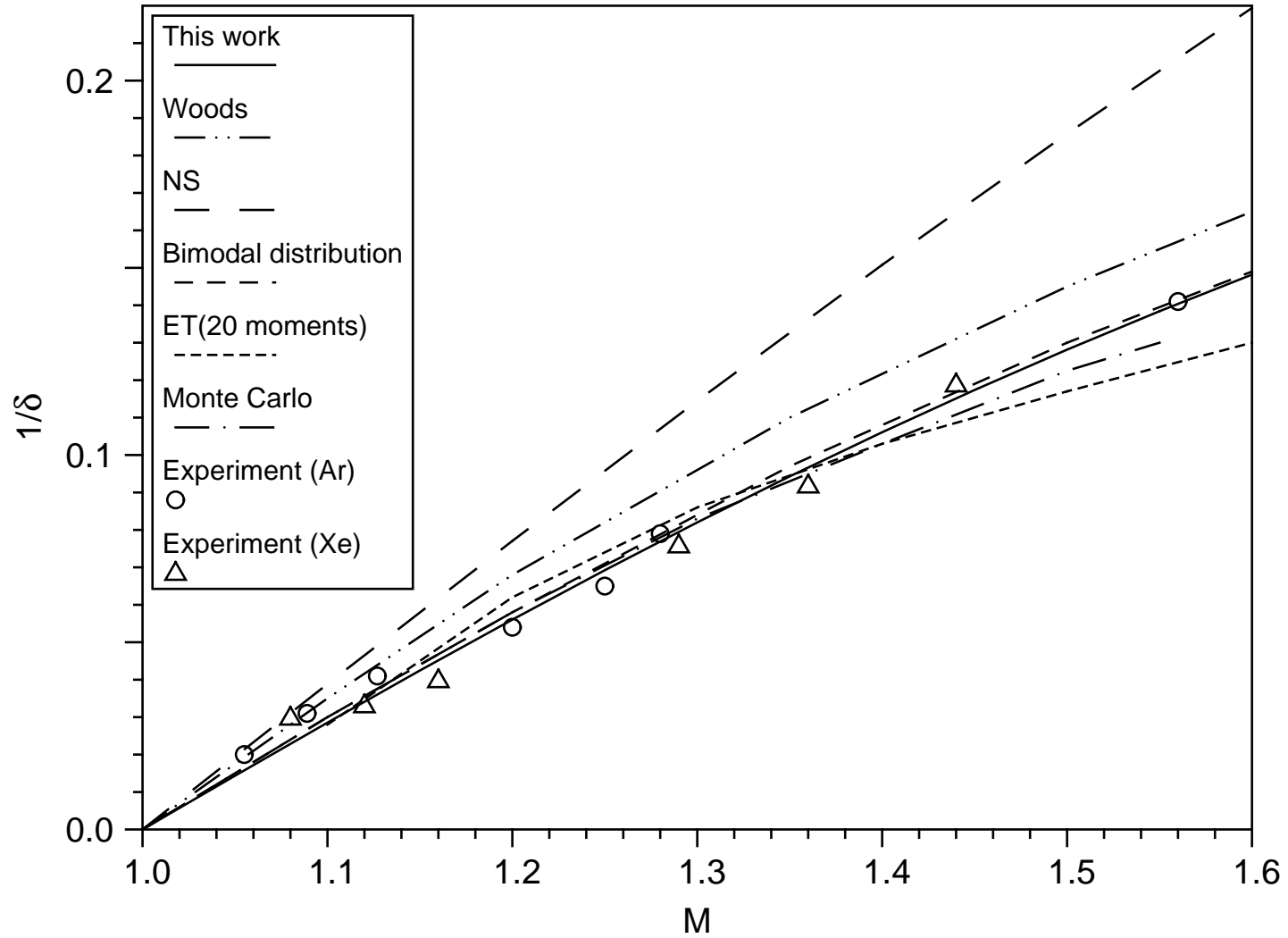
Example: Ultrasonic Sound Waves

Navier–Stokes is notoriously bad at describing sound waves when the frequency of oscillations is comparable to the free flight time of particles between collisions.

- **Linearize** equations around equilibrium: $\mathbf{u} = 0$, $T = T_0$, $\rho = \rho_0$.
- Find **dispersion relation**.
- The natural choice of Knudsen number is the frequency of the wave over the free flight time τ .



[Chen, Rao, and Spiegel, Phys. Lett. A (2000)]



[Chen, Rao, and Spiegel, Phys. Lett. A (2000)]

Conclusions and Future Work

- Want fluid equations that are valid over a wide range of Knudsen numbers.
- Modification of Chapman–Enskog method: does not use lower-order equations ([Euler](#)).
- Get “[implicit](#)” fluid equations: contain all orders in ϵ .
- [Second-order theory](#): does agreement get better?
- [Fokker–Planck](#) collision operator: application to plasmas, astrophysics, etc.
- Compare with more [experiments](#).