

Stretching and Curvature in Chaotic Mixing

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with [Allen Boozer](#) and [David Lazanja](#)

The Advection–diffusion Equation

$$\frac{\partial \phi}{\partial t} + \mathbf{v} \cdot \nabla \phi = \frac{1}{\rho} \nabla \cdot (\rho \mathbf{D} \nabla \phi)$$

- $\mathbf{v}(\mathbf{x}, t)$ — velocity field
- $\phi(\mathbf{x}, t)$ — concentration of passive scalar
- $\mathbf{D}(\mathbf{x}, t)$ — diffusivity tensor ($D/vL \ll 1$)
- $\rho(\mathbf{x}, t)$ — density

ϕ could be temperature, or the concentration of a reacting chemical, or...

Small diffusivity is the **norm** rather than the exception.

Typical values of D/vL :

Core of earth	10^{-3}
Temperature in a room	10^{-10}
Solar corona	10^{-12}
Galaxy	10^{-19}

Even a **tiny** amount of diffusivity matters.

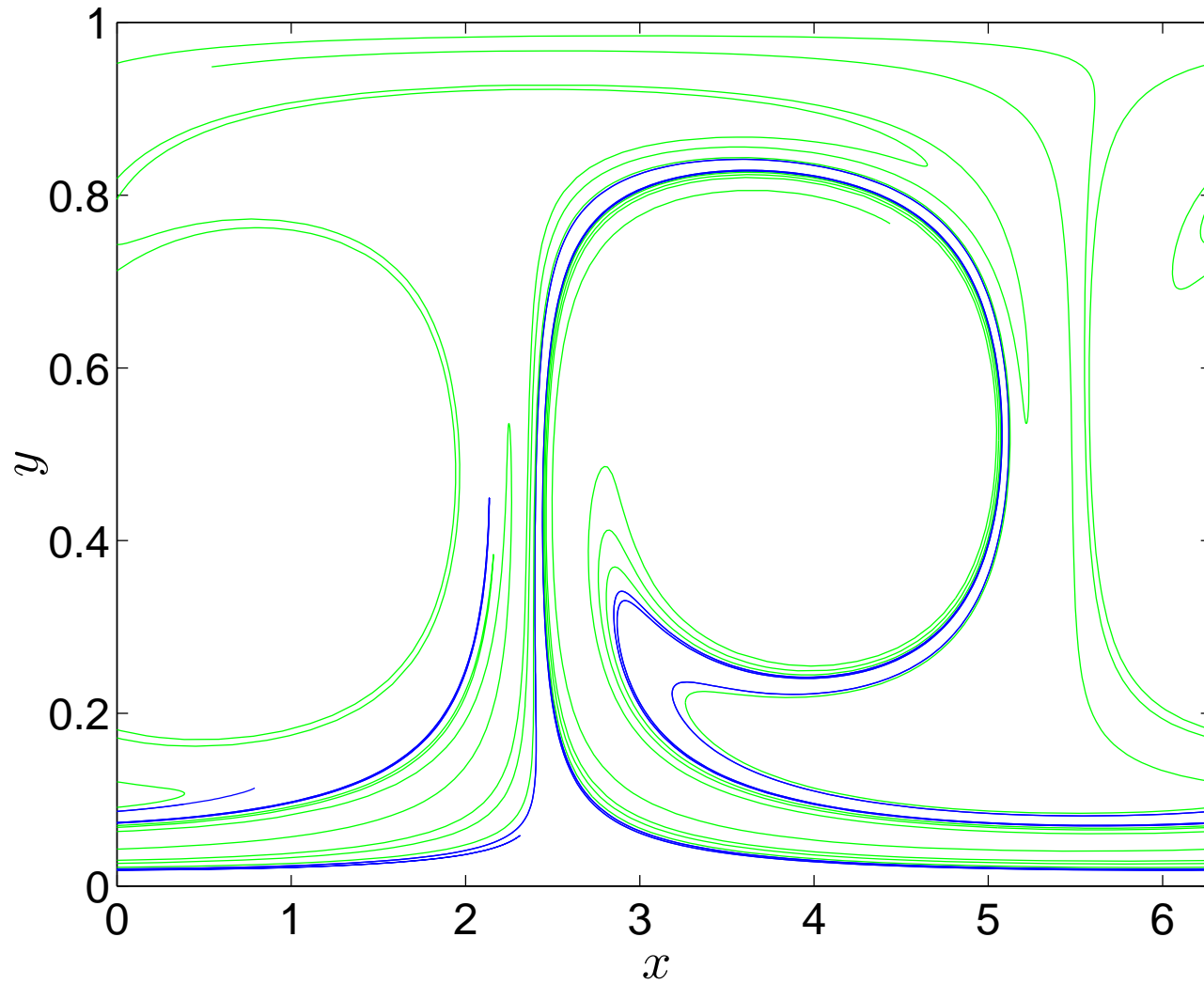
Chaotic Stirring



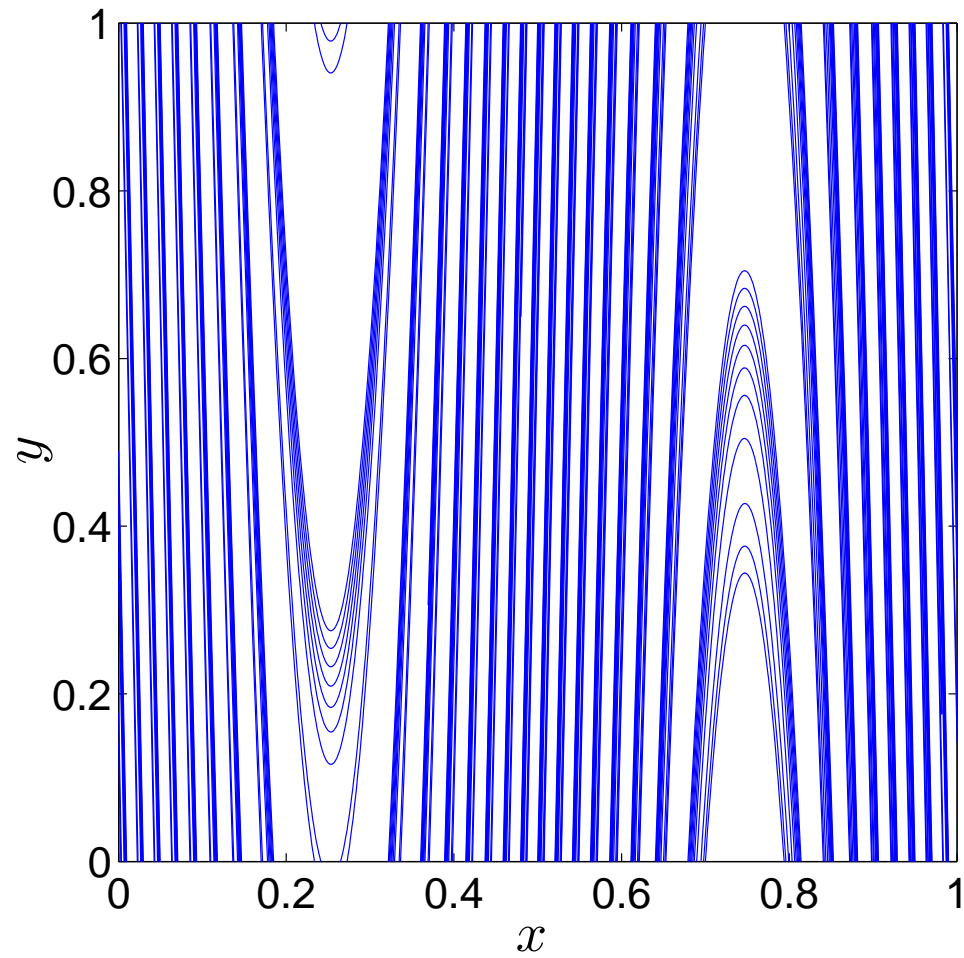
Chaotic Mixing

- Chaotic trajectories of fluid particles generates small scales, even for non-turbulent flows;
- Huge gradients of ϕ are created;
- Makes enhanced diffusion possible:
For heat in a room, turns a diffusion time of months into minutes (exponential)
- Very difficult to simulate directly : scale separation $\sim 10^{10}$;
- Important to have a clear picture of how gradients are generated...

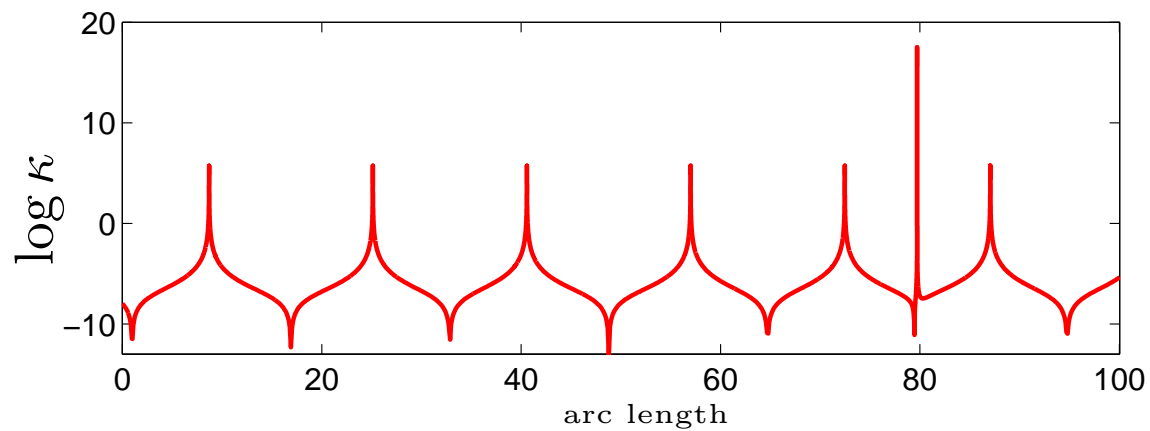
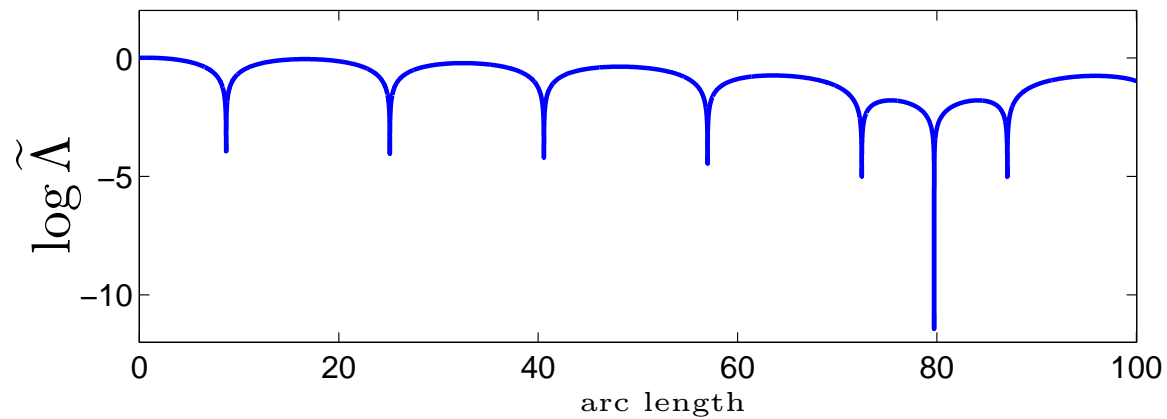
Cellular Flow



Standard Map



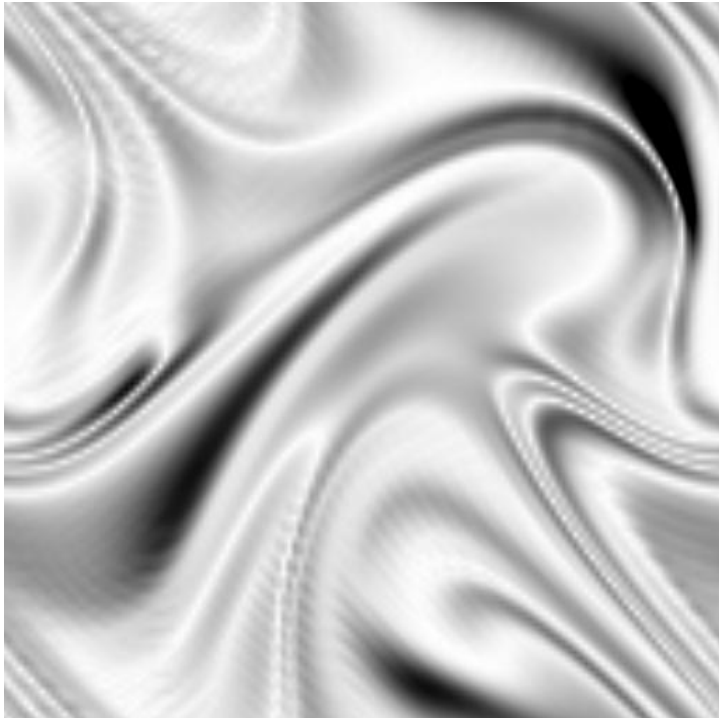
Stretching along a Material Line



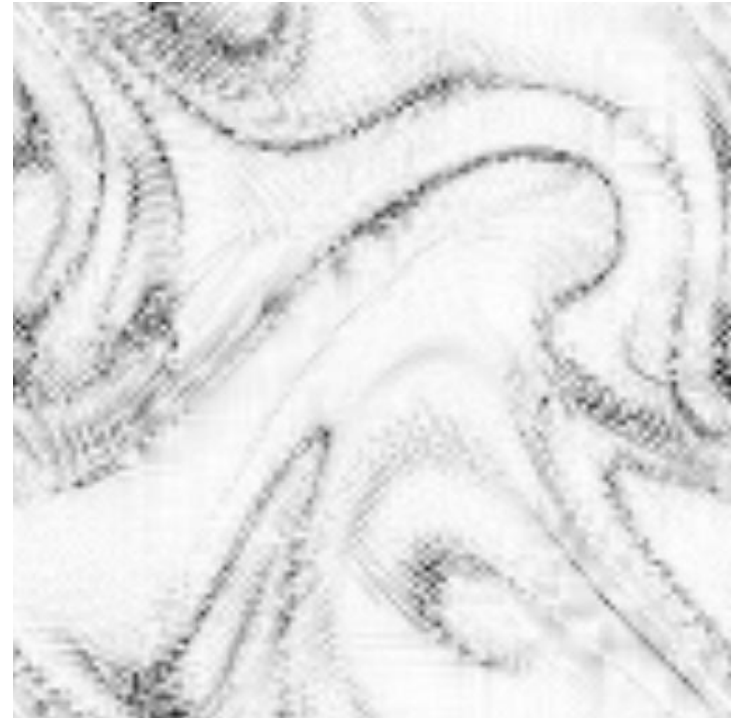
$\tilde{\Lambda}$ is small whenever the curvature is large

\Rightarrow Suppression of stretching.

Stretching and Curvature: the Dynamo



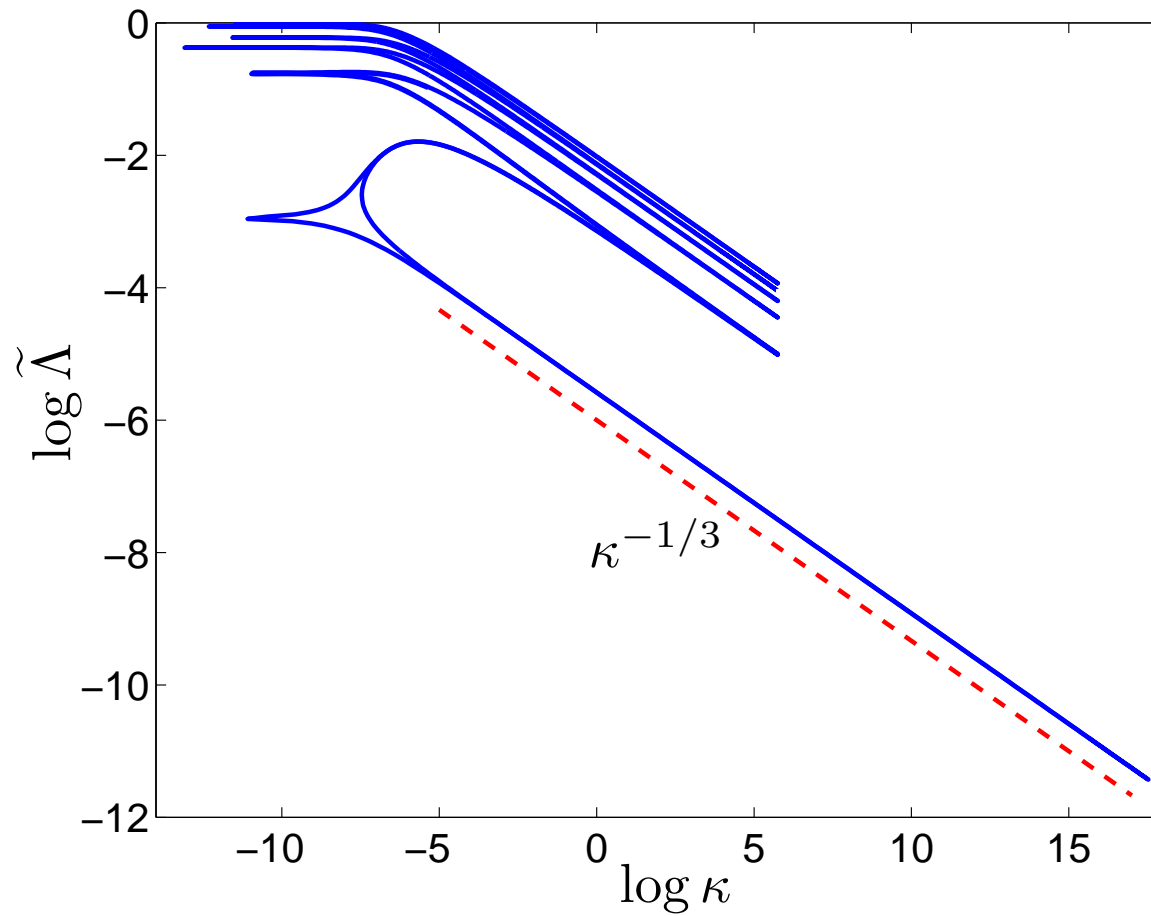
Magnetic field, B



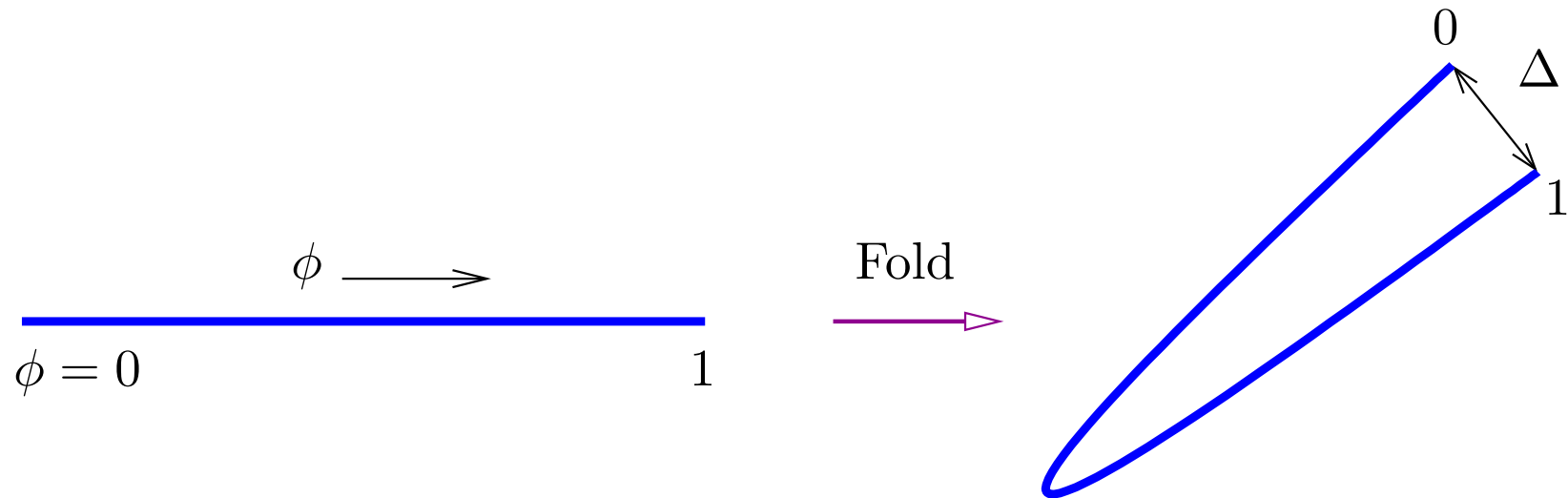
Curvature of B , κ

Schekochihin, Cowley, Maron, and Malyshkin, Phys. Rev. E (2002)

Stretching vs Curvature along a Material Line



Material Line Folded by a Flow



- Assume linear gradient of ϕ varying from 0 to 1;
- The endpoints of the line are brought to a distance Δ ;
- Enhancement in $\nabla\phi$ proportional to Δ^{-1} ;
- Points in the crest of the bend do not benefit.

A Simple Model

Consider a very sharp bend in a material line, parametrised in the x - y plane by $y = f(x)$. We Taylor expand about the minimum,

$$f(x) = \frac{1}{2}\kappa_0 x^2 + O(x^3)$$

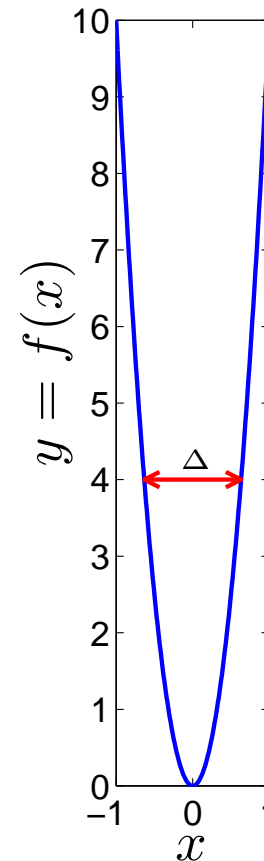
where $\kappa_0 = f''(0)$ is the curvature at the tip.

Because $f(x) \gg x$ away from the tip of the bend, we can approximate the arc length τ from $(0, 0)$ to $(x, f(x))$ by

$$\tau(x) \simeq f(x).$$

Enhancement to gradients:

$$\tilde{\Lambda}(x) = \tau(x)/x \simeq f(x)/x.$$



The curvature is $\kappa \equiv |(\hat{\mathbf{t}} \cdot \nabla)\hat{\mathbf{t}}|$, where $\hat{\mathbf{t}}$ is the unit tangent to f . To leading order this is

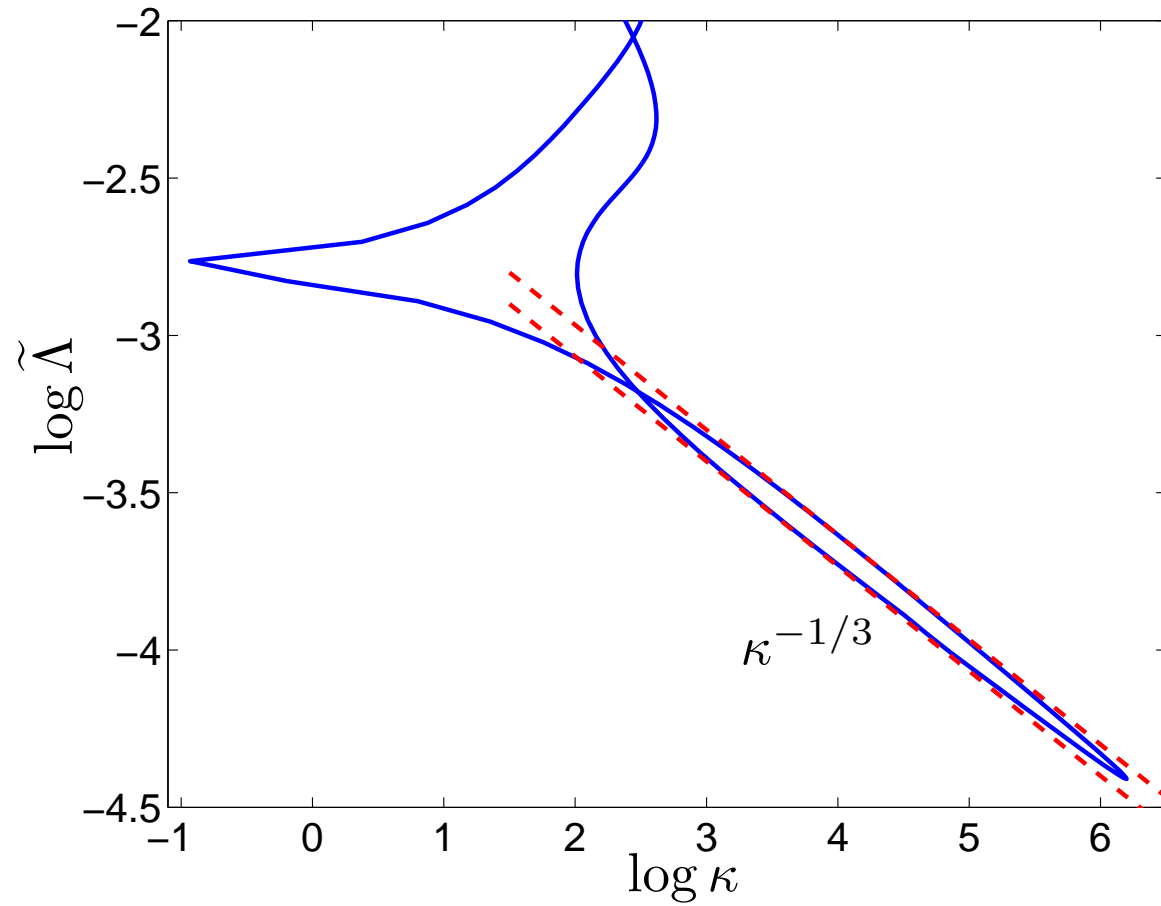
$$\kappa(x) = \kappa_0^{-2} x^{-3} + \mathcal{O}(x^{-2}), \quad \tilde{\Lambda}(x) = \kappa_0 x + \mathcal{O}(x^2).$$

Solve for x in terms of κ ,

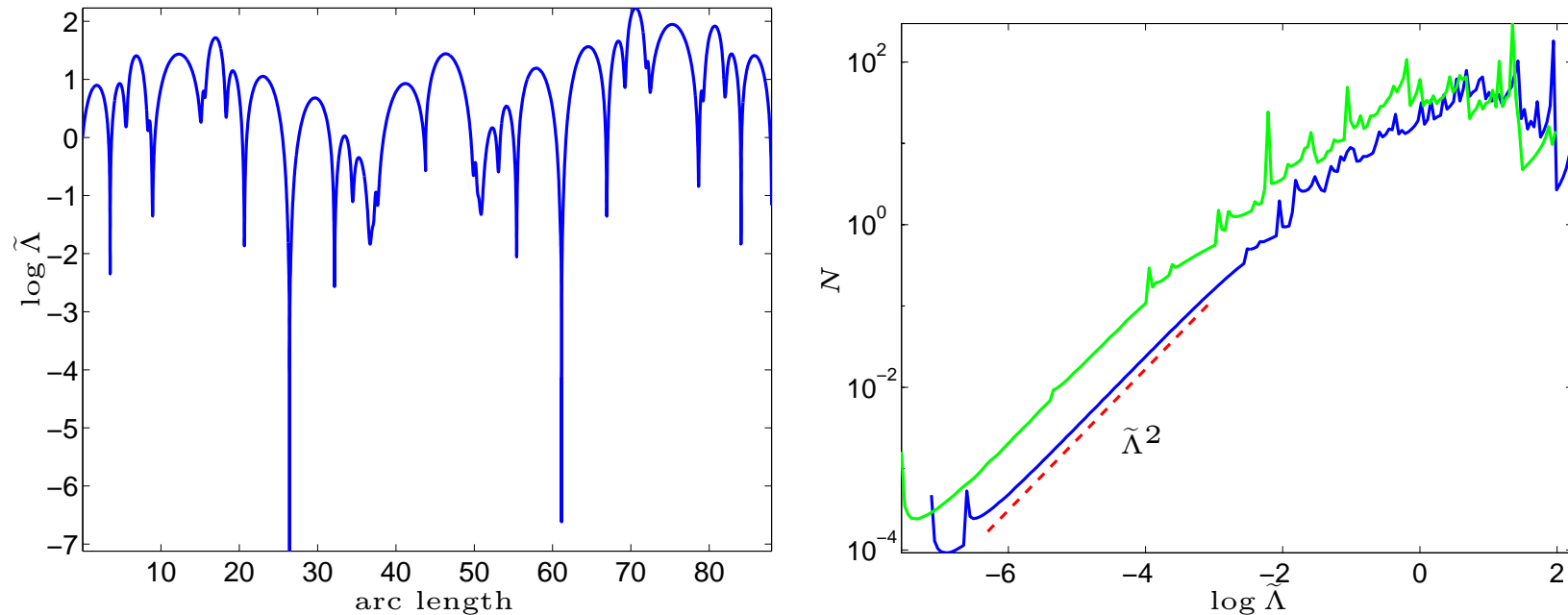
$$\tilde{\Lambda} \sim \kappa^{-1/3}$$

This simple law works remarkably well, even in flows where it is not so simple to “isolate” the folds.

Cellular Flow

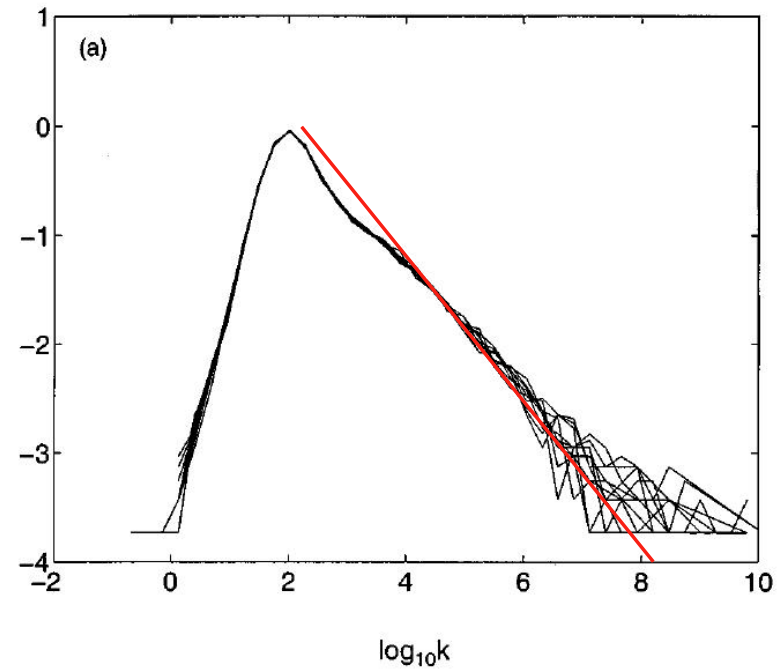
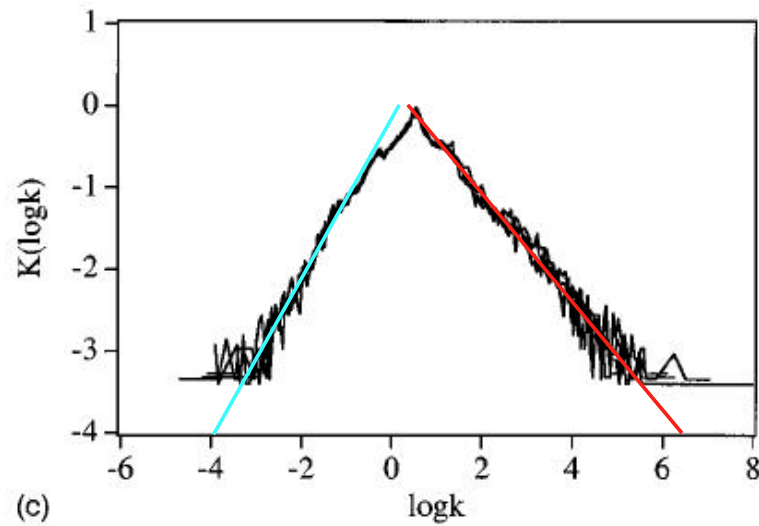


PDF of Stretching along a Material Line



The “bend” model predicts the $\tilde{\Lambda}^2$ decay of the probability of extremely low stretching events. Exponential (“fat”) tail: can have a tremendous impact on mixing.

PDF of Curvature



Stationary distribution, independent of flow.

Red line has slope $-2/3$.

Liu and Muzzio, Phys. Fluids (1996); Hobbs and Muzzio, Phys. Fluids (1998)

Summary and Other Work

- In the **small diffusivity limit**, the advection–diffusion equation cannot be solved directly because of scale separation.
- In smooth flows, important to understand the detailed manner in which gradients are enhanced through **folding**.
- Simple model captures the behaviour at **sharp bends**.
- Reduce the advection–diffusion equation to **one dimension** in Lagrangian coordinates. [Thiffeault, *Physics Letters A* (2001)]
- Requires the use of **differential constraints** [Thiffeault and Boozer, *Chaos* (2001); Thiffeault, *Nonlinearity* (2001)].
- Breakup of oil droplets (with T. Solomon), open flows, chemical mixing. . .