Large-scale Eigenfunctions and Mixing

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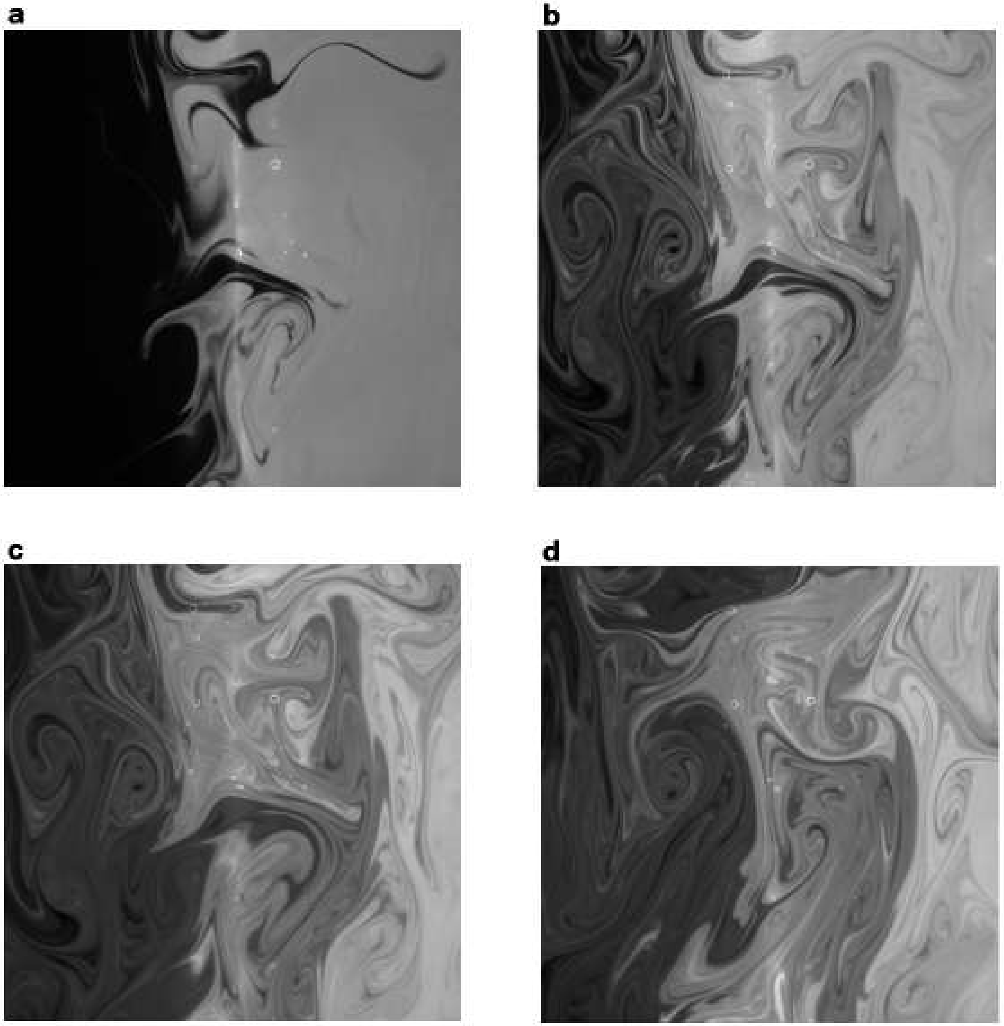
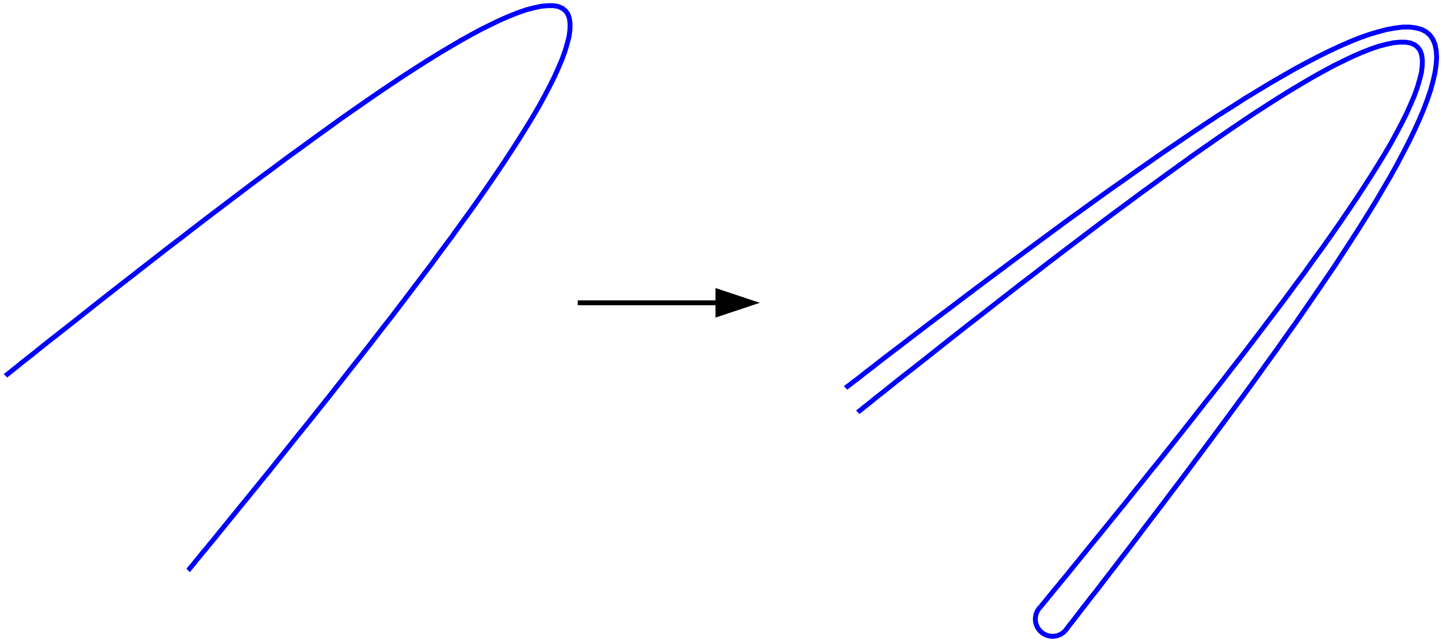
Persistent Pattern

Disordered array ($i = 2, 20, 50, 50.5$)



[Rothstein, Henry, and Gollub, Nature **401**, 770 (1999)]

Evolution of Pattern



- “Striations”
- Smoothed by diffusion
- Eventually settles into “pattern” (eigenfunction)

Local vs Global Regimes of Mixing

Local theory:

- Based on distribution of Lyapunov exponents.

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- Eigenfunction of advection–diffusion operator.
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[Pierrehumbert, Chaos Sol. Frac. (1994)]	Strange eigenmode
[Fereday et al., Wonhas and Vassilicos, PRE (2002)]	Baker's map
[Sukhatme and Pierrehumbert, PRE (2002)]	
[Fereday and Haynes (2003)]	Unified description
[Thiffeault and Childress (2003)]	Modified Cat map

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Global theory:

- Eigenfunction of advection–diffusion operator.
- Map allows analytical results.

The Map

We consider a diffeomorphism of the 2-torus $\mathbb{T}^2 = [0,1]^2$,

$$\mathcal{M}(\boldsymbol{x}) = \mathbb{M} \cdot \boldsymbol{x} + \phi(\boldsymbol{x}),$$

where

$$\mathbb{M} = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}; \qquad \phi(\boldsymbol{x}) = \frac{K}{2\pi} \begin{pmatrix} \sin 2\pi x_1 \\ \sin 2\pi x_1 \end{pmatrix};$$

$\mathbb{M} \cdot \boldsymbol{x}$ is the Arnold cat map.

The map $\mathcal{M}$ is area-preserving and chaotic.

For $K = 0$ the stretching of fluid elements is homogeneous in space.

Advection and Diffusion

Iterate the map and apply the heat operator to a scalar field (which we call temperature for concreteness) distribution $\theta^{(i-1)}(\boldsymbol{x})$,

$$\theta^{(i)}(\boldsymbol{x}) = \mathcal{H}_\epsilon\,\theta^{(i-1)}(\mathcal{M}^{-1}(\boldsymbol{x}))$$

where ϵ is the diffusivity, with the heat operator $\mathcal{H}_\epsilon$. In other words: advect instantaneously and then diffuse for one unit of time.
After Fourier expanding $\theta^{(i)}(\boldsymbol{x})$, the effect of advection and diffusion becomes

$$\hat{\theta}^{(i)}(\boldsymbol{x}) = \sum_{\boldsymbol{q}} \mathbb{T}_{\boldsymbol{k}\boldsymbol{q}}\,\hat{\theta}^{(i-1)}_{\boldsymbol{q}}\,,$$

with the transfer matrix,

$$\mathbb{T}_{\boldsymbol{k}\boldsymbol{q}} = \mathrm{e}^{-\epsilon\,\boldsymbol{q}^2}\,\delta_{0,Q_2}\,\mathrm{i}^{Q_1}\,J_{Q_1}\left((k_1+k_2)\,K\right), \qquad \boldsymbol{Q} := \boldsymbol{k}\cdot\mathbb{M} - \boldsymbol{q}.$$

Variance: A measure of mixing

In the absence of diffusion ($\epsilon = 0$), the variance $\sigma^{(i)}$

$$\sigma^{(i)} := \int_{\mathbb{T}^2} \left|\theta^{(i)}(\boldsymbol{x})\right|^2 \mathrm{d}\boldsymbol{x} = \sum_{\boldsymbol{k}} \sigma^{(i)}_{\boldsymbol{k}}, \qquad \sigma^{(i)}_{\boldsymbol{k}} := \left|\hat{\theta}^{(i)}_{\boldsymbol{k}}\right|^2$$

is preserved. (We assume the spatial mean of θ is zero.)

For $\epsilon > 0$ the variance decays.

We consider the case $\epsilon \ll 1$, of greatest practical interest.

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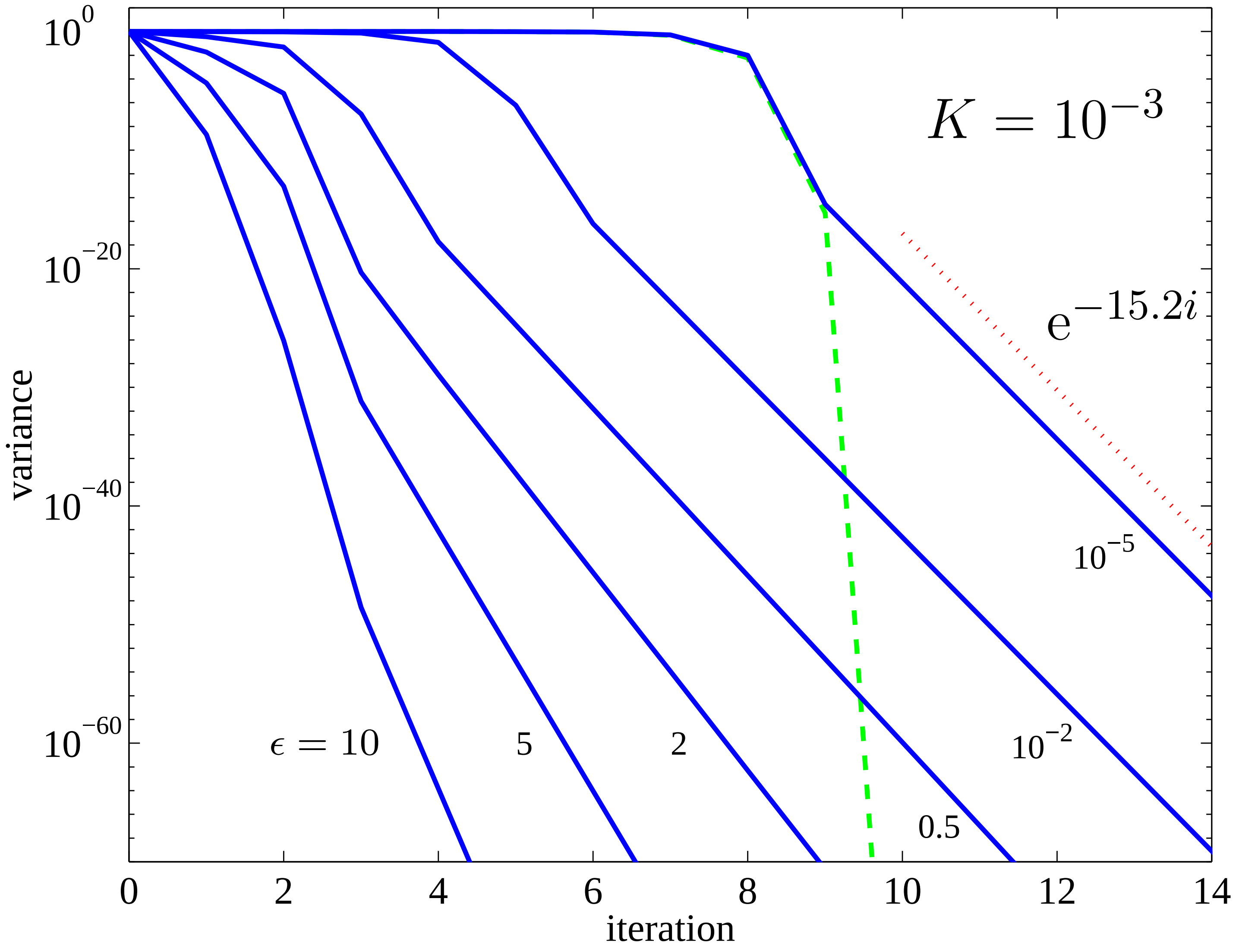
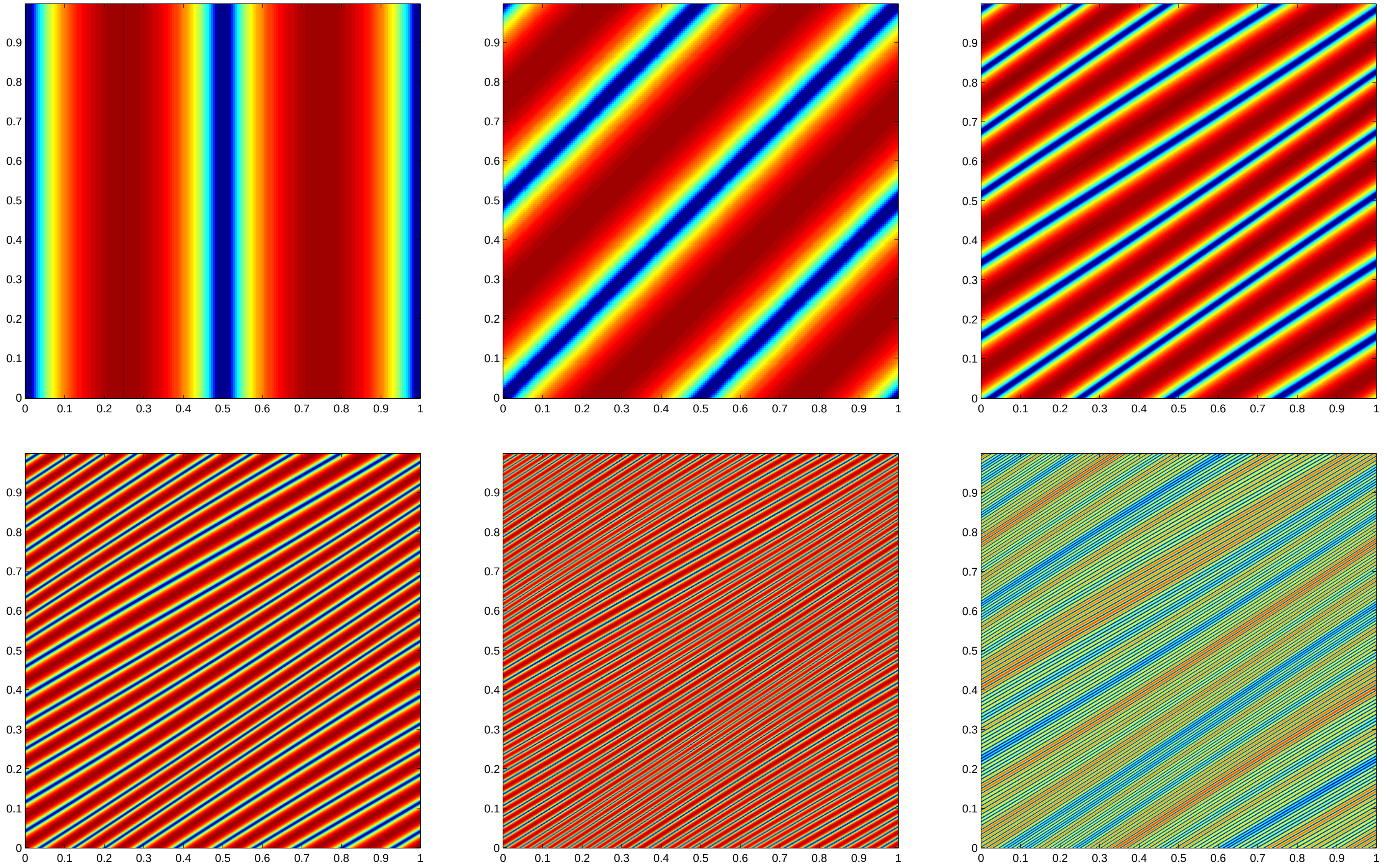
- $\theta^{(i)}$ settles into an eigenfunction of the A–D operator that sets the exponential decay rate.

Decay of Variance

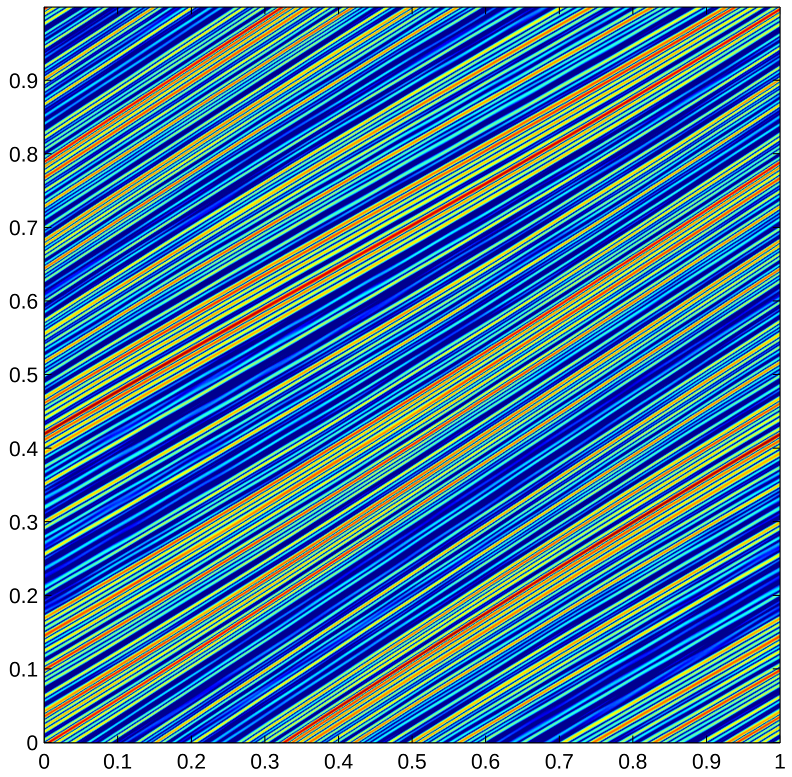


Variance: 5 iterations for $K = 0.3$ and $\epsilon = 10^{-3}$

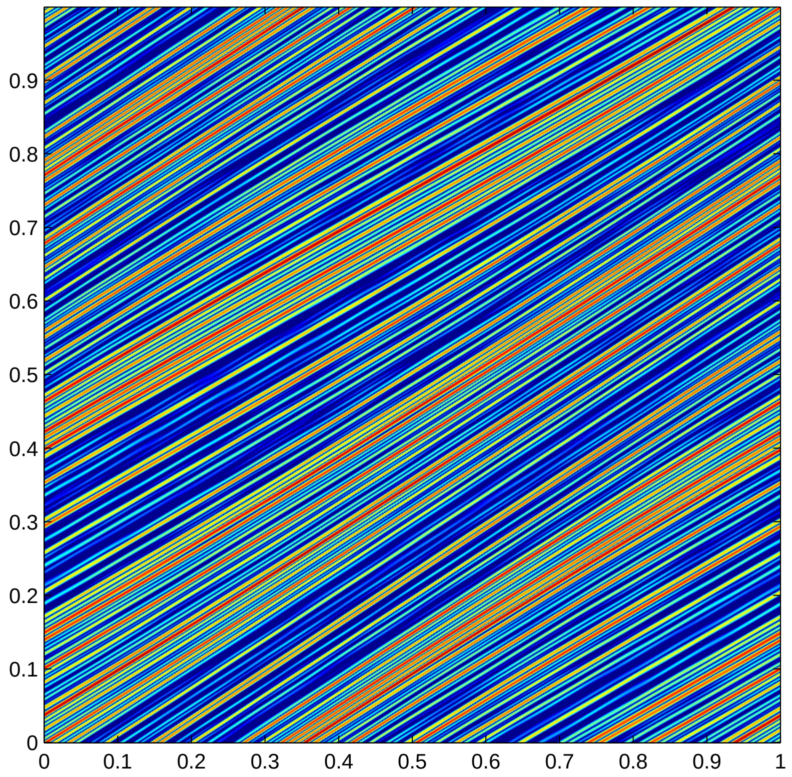


Eigenfunction for $K = 0.3$ and $\epsilon = 10^{-3}$

(Renormalised by decay rate)

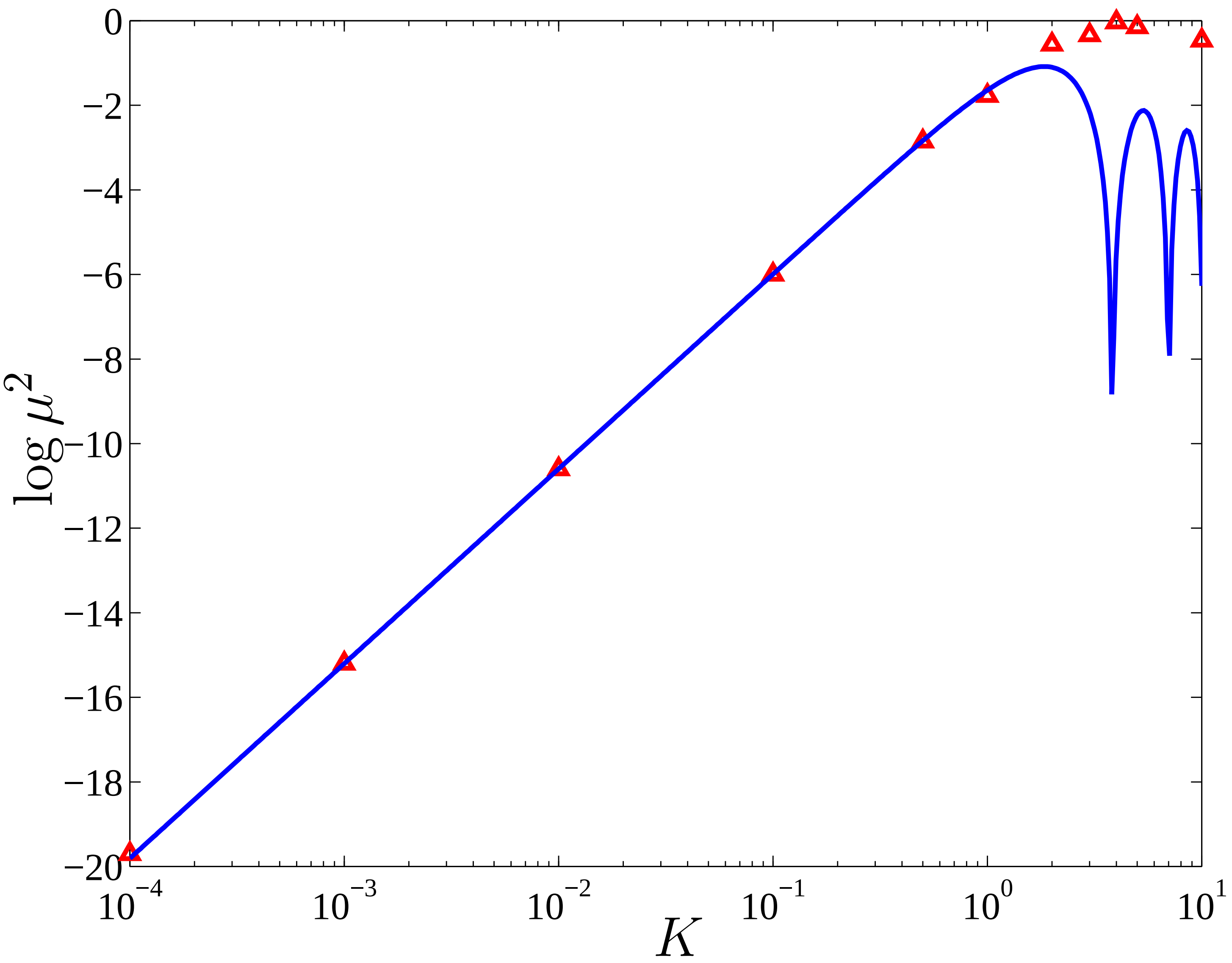


$i = 25$



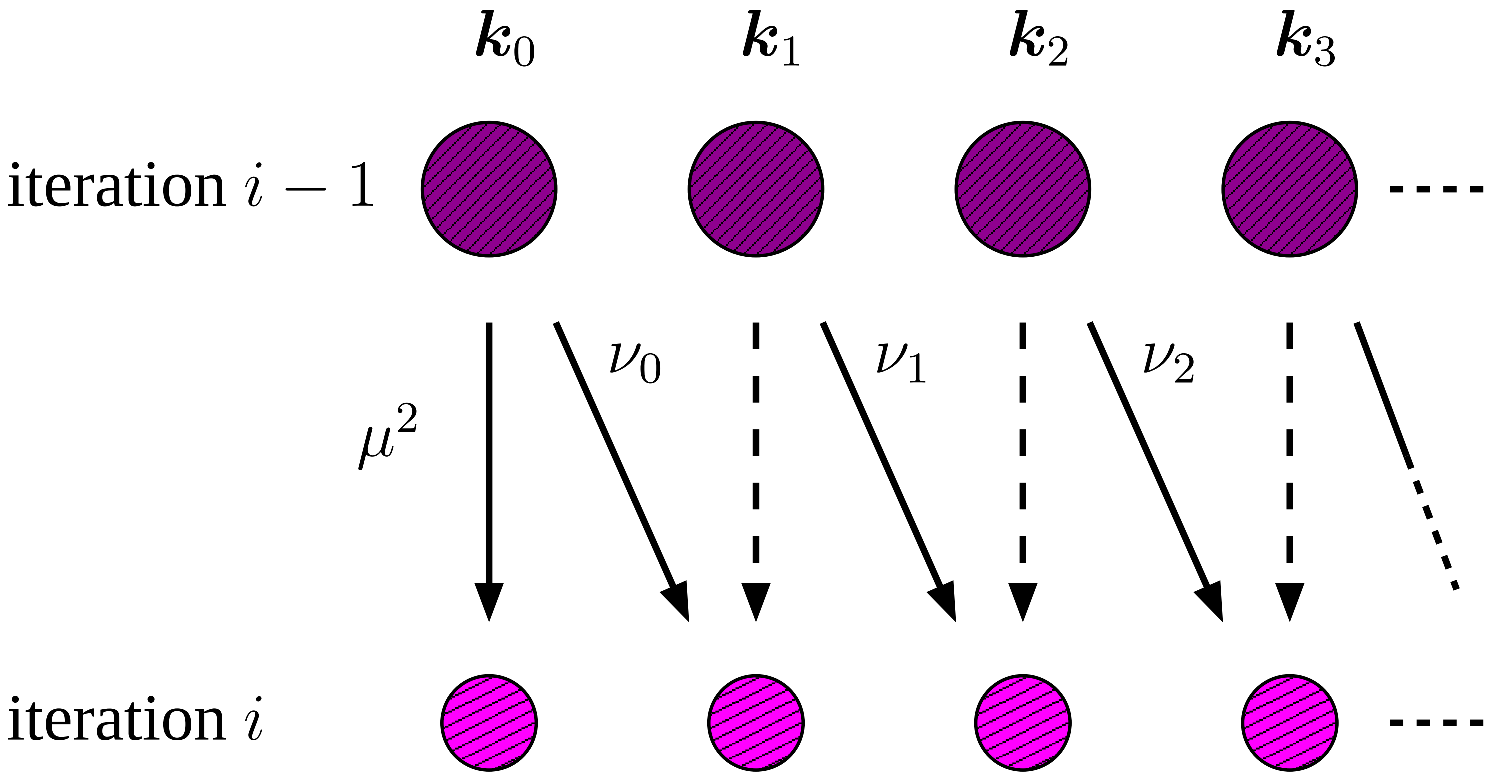
$i = 30$

Decay Rate as $\epsilon \to 0$

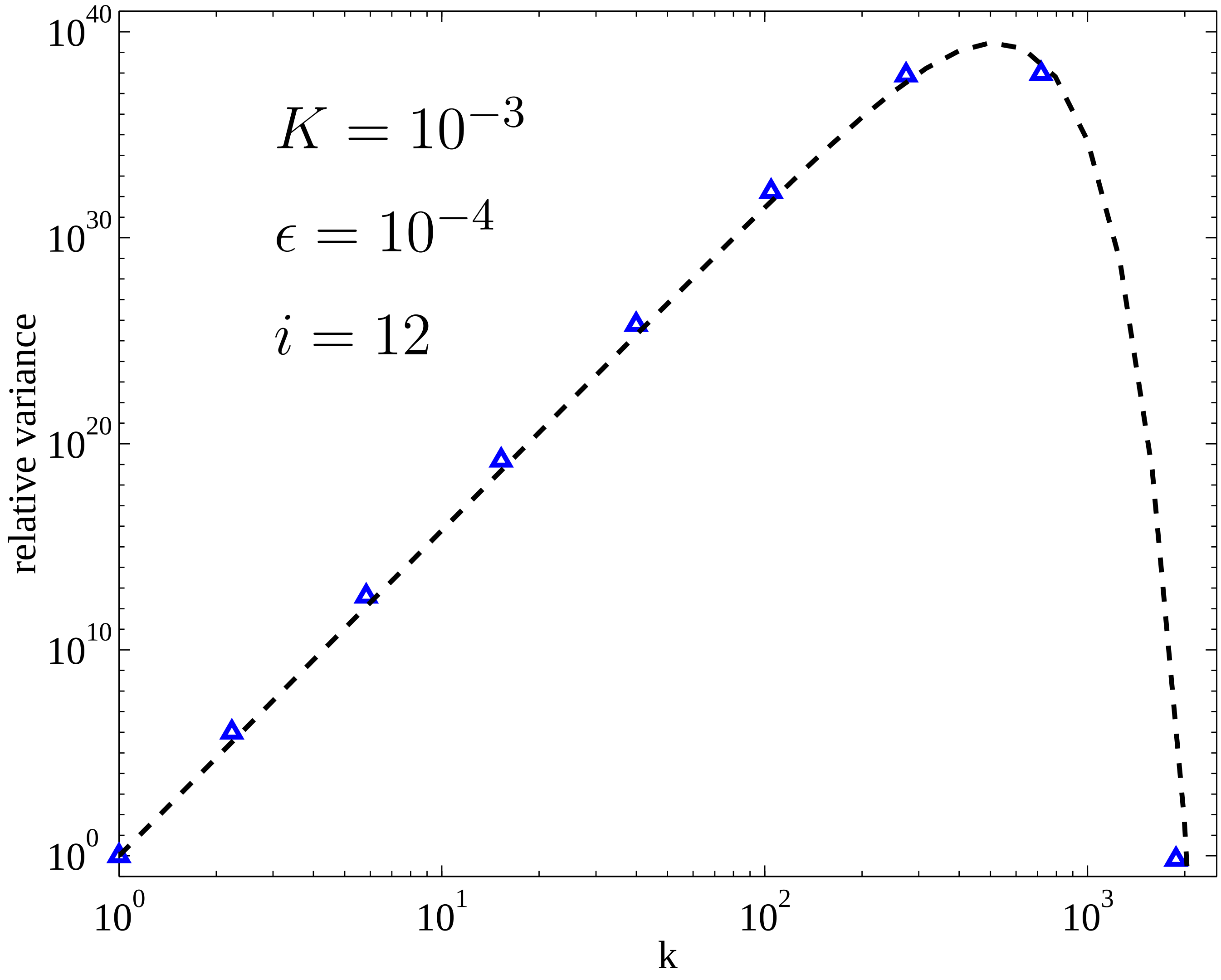


Eigenfunction: One Iteration

The wavenumbers are mapped back to themselves, with their variance decreased by a uniform factor $\mu^2 < 1$ (vertical arrows). But at the same time the modes are mapped to next one down the cascade following the diagonal arrows.



Spectrum of Variance



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