Chaotic Mixing in a Torus Map

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Regular array of magnets

[Rothstein, Henry, and Gollub, Nature 401, 770 (1999)]
Persistent Pattern

Disordered array \((i = 2, 20, 50, 50.5)\)
Local vs Global Regimes of Mixing

Local theory:

- Based on distribution of Lyapunov exponents.
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- Applies if initial scale of concentration small.

Global theory:

- Eigenfunction of advection–diffusion operator.
- Applies if initial scale large, or for inverse cascade.

Today: Focus on Global theory.
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[Fereday et al., Wonhas and Vassilicos, PRE (2002)]
[Sukhatme and Pierrehumbert, PRE (2002)]

Average over angles
Statistical model
Strange eigenmode
Baker’s map
Unified description
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- Today: Focus on Global theory.
- Map allows analytical results (enough said!).
The Map

We consider a diffeomorphism of the 2-torus $\mathbb{T}^2 = [0, 1]^2$,

$$\mathcal{M}(\mathbf{x}) = \mathbf{M} \cdot \mathbf{x} + \phi(\mathbf{x}),$$

where

$$\mathbf{M} = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}; \quad \phi(\mathbf{x}) = \frac{K}{2\pi} \begin{pmatrix} \sin 2\pi x_1 \\ \sin 2\pi x_1 \end{pmatrix};$$

$\mathbf{M} \cdot \mathbf{x}$ is the Arnold cat map.

The map $\mathcal{M}$ is area-preserving and chaotic.

For $K = 0$ the stretching of phase-space elements is uniform in space (homogeneous). For small $K$ it is hyperbolic.
Advection and Diffusion

Iterate the map and apply the heat operator to a scalar field (which we call temperature for concreteness) distribution \( \theta^{(i-1)}(\mathbf{x}) \),

\[
\theta^{(i)}(\mathbf{x}) = \mathcal{H}_\epsilon \theta^{(i-1)}(\mathcal{M}^{-1}(\mathbf{x}))
\]

where \( \epsilon \) is the diffusivity, with the heat operator \( \mathcal{H}_\epsilon \) and kernel \( h_\epsilon \)

\[
\mathcal{H}_\epsilon \theta(\mathbf{x}) := \int_{\mathbb{T}^2} h_\epsilon(\mathbf{x} - \mathbf{y}) \theta(\mathbf{y}) \, d\mathbf{y};
\]

\[
h_\epsilon(\mathbf{x}) = \sum_k \exp(2\pi i \mathbf{k} \cdot \mathbf{x} - \mathbf{k}^2 \epsilon).
\]

In other words: advect instantaneously and then diffuse for one unit of time.
Fourier expand $\theta^{(i)}(x)$,

$$\theta^{(i)}(x) = \sum_k \hat{\theta}_k^{(i)} e^{2\pi i k \cdot x}.$$  

The effect of advection and diffusion becomes

$$\hat{\theta}^{(i)}(x) = \sum_q T_{kq} \hat{\theta}_q^{(i-1)},$$  

with the transfer matrix,

$$T_{kq} := \int_{\mathbb{T}^2} \exp \left( 2\pi i (q \cdot x - k \cdot \mathcal{M}(x)) - \epsilon q^2 \right) \, dx,$$

$$= e^{-\epsilon q^2} \delta_{0,Q_2} i^{Q_1} J_{Q_1} \left( (k_1 + k_2) K \right), \quad Q := k \cdot \mathcal{M} - q,$$

where the $J_Q$ are the Bessel functions of the first kind.
Variance: A measure of mixing

In the absence of diffusion ($\epsilon = 0$), the variance $\sigma^{(i)}$

$$\sigma^{(i)} := \int_{T^2} |\theta^{(i)}(\mathbf{x})|^2 \, d\mathbf{x} = \sum_k \sigma_k^{(i)}, \quad \sigma_k^{(i)} := |\hat{\theta}_k^{(i)}|^2$$

is preserved. (We assume the spatial mean of $\theta$ is zero.) For $\epsilon > 0$ the variance decays.

We consider the case $\epsilon \ll 1$, of greatest practical interest.
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Three phases:

- The variance is initially constant;
- It then undergoes a rapid superexponential decay;
- $\theta^{(i)}$ settles into an eigenfunction of the A–D operator that sets the exponential decay rate.
Decay of Variance

$K = 10^{-3}$

$e^{-15.2i}$

$10^{-5}$

$10^{-2}$

$0.5$

$10^0$

$10^{-20}$

$10^{-40}$

$10^{-60}$

iteration

$\epsilon = 10$

$5$

$2$

$0$

$2$

$4$

$6$

$8$

$10$

$12$

$14$
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Constant (Stirring) Phase

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- However, there is a cascade of the variance to larger wavenumbers under the action of $M^{-1}$ in the map. (Neglect $K$ term.)
- This is the well-known “filamentation” effect in chaotic flows: the stretching and folding action of the flow causes rapid variation of the temperature across the folds.
- Can no longer neglect diffusion after a number of iterations

$$i_1 \simeq 1 + \left( \log \epsilon^{-1} / \log \Lambda^2 \right) \simeq 6 \quad \text{for } \epsilon = 10^{-5},$$

where $\Lambda = (3 + \sqrt{5})/2$ is the largest eigenvalue of $M^{-1}$. 
Variance: 5 iterations for $K = 0.3$ and $\epsilon = 10^{-3}$
For small $K$ and $k$, we have $J_0 ((k_1 + k_2) K) \gg J_1 ((k_1 + k_2) K)$, so we set $K = 0$ and retain only the $Q_1 = 0$ term in the transfer matrix,

$$\mathbb{T}_{kq} = e^{-\epsilon q^2} \delta_{0,Q} + O\left( (k_1 + k_2)^2 K^2 \right);$$

The nonvanishing matrix elements of $\mathbb{T}$ have $k = q \cdot M^{-1}$. 
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If initially the variance is concentrated in a single wavenumber $q_0$, then after one iteration it will all be in $q_0 \cdot \mathbb{M}^{-1}$, after two in $q_0 \cdot \mathbb{M}^{-2}$, etc.

The length of $q$ is multiplied by $\Lambda$ at each iteration.
Superexponential Phase

For small $K$ and $\mathbf{k}$, we have $J_0 ((k_1 + k_2)K) \gg J_1 ((k_1 + k_2)K)$, so we set $K = 0$ and retain only the $Q_1 = 0$ term in the transfer matrix,

$$\mathbb{T}_{kq} = e^{-\epsilon q^2} \delta_{0,Q} + \mathcal{O}((k_1 + k_2)^2 K^2);$$

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The length of $q$ is multiplied by $\Lambda$ at each iteration.

But each at each step the variance is multiplied by the diffusive decay factor $\exp(-\epsilon q^2)$, with $q$ getting exponentially larger; the net decay is thus superexponential.
Exponential Phase

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- This direct cascade process dominates at first, but it is so efficient that eventually we must examine the effect of the the wave term ($\sin$), which is felt through the higher-order Bessel functions in the transfer matrix.
- Can the wave term lead to the formation of an eigenfunction of the advection–diffusion operator, which would imply exponential decay?
Recall:

\[ T_{kq} = e^{-\epsilon q^2} \delta_{0,Q_2} i^{Q_1} J_{Q_1} ( (k_1 + k_2) K ), \quad Q := k \cdot \mathbb{M} - q, \]

Consider a matrix element for which \( Q_1 \neq 0 \). This means that the initial \((q)\) and final \((k)\) wavenumbers connected by that matrix element can differ from \( k \cdot \mathbb{M} = q \) by \( Q_1 \) in their first component.
An Eigenfunction?

Recall:

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Consider a matrix element for which $Q_1 \neq 0$. This means that the initial $(q)$ and final $(k)$ wavenumbers connected by that matrix element can differ from $k \cdot M = q$ by $Q_1$ in their first component.

Is it possible for a wavenumber to be mapped back onto itself by such a coupling? Seek solutions to

$$\begin{pmatrix} q_1 & q_2 \end{pmatrix} \cdot M = \begin{pmatrix} q_1 + Q_1 & q_2 \end{pmatrix} \quad \implies \quad \begin{pmatrix} q_1 & q_2 \end{pmatrix} = \begin{pmatrix} 0 & Q_1 \end{pmatrix}.$$

The matrix element connecting the $(0 \ Q_1)$ mode to itself is

$$T_{(0 \ Q_1),(0 \ Q_1)} = e^{-\epsilon Q_1^2} i^{Q_1} J_{Q_1} (Q_1 K).$$
Eigenfunction for $K = 0.3$ and $\epsilon = 10^{-3}$

(Renormalised by decay rate)

$i = 25$

$i = 30$
Decay Rate

For small $K$, the dominant Bessel function is $J_1$, so the decay factor $\mu^2$ for the variance is given by

$$\mu = |T_{(0\ 1),(0\ 1)}| = e^{-\epsilon} J_1(K) = \frac{1}{2} K + O(\epsilon K, K^2).$$

Hence, for small $K$ the decay rate is limited by the $(0\ 1)$ mode. The decay rate is independent of $\epsilon$ for $\epsilon \to 0$. 

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This is because in the baker’s map the discontinuity generates many slowly-decaying harmonics at each step.
Decay Rate as $\epsilon \to 0$
Transition from Superexponential to Exponential

- Now that the mechanism of exponential decay is understood, we can go back and describe the condition for breakdown of the superexponential solution.
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The superexponential phase thus ends when the variance at large wavenumbers equals that in mode \( (0 1) \).

Assuming that the variance resides entirely in the \( k_0 = (0 1) \) mode initially, the condition for breakdown is

\[
\mu^i_2 = \exp \left( -\epsilon \| k_0 \cdot M^{-(i_2 - 1)} \|^2 \right),
\]

where \( \mu^2 \) is the decay factor of the variance in \( k_0 \).
After substituting \( k_0 \cdot M^{-(i_2-1)} \| \sim \Lambda^{i_2-1} \), solve numerically for \( i_2 \).

For \( K = 10^{-3} \) and \( \epsilon = 10^{-5} \), we have \( i_2 \simeq 9.2 \), numerical results.
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For $K = 10^{-3}$ and $\epsilon = 10^{-5}$, we have $i_2 \simeq 9.2$, numerical results.

For $\epsilon \ll 1$, approximate solution given by

$$i_2 \simeq 1 + \log \left( \epsilon^{-1} \log \mu^{-1} \right) / \log \Lambda^2,$$

which gives $i_2 \simeq 8$ for $K = 10^{-3}$, $\epsilon = 10^{-5}$. 
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Subtracting $i_1 = 1 + \log \epsilon^{-1} / \log \Lambda^2$, the onset of the superexponential phase, we find the duration of the phase is

$$i_2 - i_1 \simeq \frac{\log \log \mu^{-1}}{\log \Lambda^2}.$$
Duration of Superexponential Phase

\[ i_2 - i_1 \approx \frac{\log \log \mu^{-1}}{\log \Lambda^2} \]

- Independent of \( \epsilon \);
Duration of Superexponential Phase

\[ i_2 - i_1 \simeq \frac{\log \log \mu^{-1}}{\log \Lambda^2} \]

- Independent of \( \epsilon \);
- Very weak dependence on the decay rate \( \log \mu \);
Duration of Superexponential Phase

\[ \dot{i}_2 - \dot{i}_1 \approx \frac{\log \log \mu^{-1}}{\log \Lambda^2} \]

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- Unless \( \mu \) is small (recall that \( 0 < \mu < 1 \)), the superexponential phase is short and may not be noticeable at all, resembling instead a smooth transition;
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- For \( \log \mu^{-1} > 1 \) there is no superexponential phase at all;
- Observed in experiments? There \( \mu \) tends to be closer to unity, so unlikely. But...
• The long-wavelength mode $(0, 1)$ is the bottleneck that determines the decay rate, for small $K$. 
Variance Spectrum of the Eigenfunction

- The long-wavelength mode \((0 \ 1)\) is the bottleneck that determines the decay rate, for small \(K\).
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But this dominant mode does not determine the structure of
the eigenfunction.

In fact, a very small amount of the total variance actually
resides in that bottleneck mode: the variance is concentrated
at small scales.
The variance is taken out of the $(0 \ 1)$ mode by the map: there is a cascade to larger wavenumber through the action of $M^{-1}$:

$$(0 \ 1) \rightarrow (-1 \ 2) \rightarrow (-3 \ 5) \rightarrow (-8 \ 13) \rightarrow \ldots.$$ 

These become more and more aligned with the stable (contracting) direction of the map.
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The amplitude of the wavenumbers is multiplied at each step by a factor $\Lambda = (3 + \sqrt{5})/2 \approx 2.618$, the largest eigenvalue of $M^{-1}$. 

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**Cascade**

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But we have seen that the exponential decay rate suggests that the scalar concentration is in an eigenfunction of the advection–diffusion operator.

What is going on?
The wavenumbers are mapped back to themselves, with their variance decreased by a uniform factor $\mu^2 < 1$ (vertical arrows). But at the same time the modes are mapped to next one down the cascade following the diagonal arrows.
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If we denote by \( \sigma_n^{(i)} := |\hat{\theta}_{k_n}|^2 \) the variance in mode \( k_n \) at the \( i \)th iteration, we have

\[
\sigma_n^{(i)} = \mu^2 \sigma_n^{(i-1)}, \quad n = 0, 1, \ldots,
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\sigma_n^{(i)} = \nu_{n-1} \sigma_{n-1}^{(i-1)}, \quad n = 1, 2, \ldots.
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\end{align*}
\]

These two recurrences can be combined to give

\[
\Sigma_n^{(i)} := \frac{\sigma_n^{(i)}}{\sigma_0^{(i)}} = \frac{\nu_{n-1} \nu_{n-2} \cdots \nu_0}{\mu^{2n}} = \mu^{-2n} \exp\left(-2\epsilon \sum_{m=0}^{n-1} k_m^2\right),
\]

where \( \Sigma_n^{(i)} \) is the relative variance in the \( n \)th mode.
The wavenumber is given by the exponential recursion,

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\[ \| \mathbf{k}_n \| \simeq \Lambda \| \mathbf{k}_{n-1} \| \implies \| \mathbf{k}_n \| \simeq \Lambda^n \| \mathbf{k}_0 \| = \Lambda^n. \]

Solve for \( n = \log \| \mathbf{k}_n \| / \log \Lambda \) and rewrite the relative variance as

\[ \sum_n^{(i)} \simeq \| \mathbf{k}_n \|^{-2 \log \mu / \log \Lambda} \exp \left( -2 \epsilon \frac{k_n^2}{\Lambda^2} \right), \]

where we retained only the \( k_{n-1}^2 \) (last) term of the sum.
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where we retained only the \( k_{n-1}^2 \) (last) term of the sum.

Does not (and should not) depend on the iteration number, \( i \), and depends only on \( n \) through \( k_n \). Find

\[ \Sigma(k) = k^{2\zeta} \exp \left( -2\epsilon k^2 / \Lambda^2 \right), \quad \zeta := -\log \mu / \log \Lambda, \]

the spectrum of relative variance.
Spectrum of Variance

\[ K = 10^{-3} \]
\[ \epsilon = 10^{-4} \]
\[ i = 12 \]
\[ \Sigma(k) = k^{2\zeta} \exp \left( -2\epsilon k^2 / \Lambda^2 \right) , \quad \zeta := -\log \mu / \log \Lambda \]

- Since \( \mu < 1 \) and \( \Lambda > 1 \), we have \( \zeta > 0 \).
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- This implies that there is more variance at the large wavenumbers than at the slowest-decaying mode \( k_0 \).
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- This implies that there is more variance at the large wavenumbers than at the slowest-decaying mode \( k_0 \).

Find the maximum of \( \Sigma(k) \),

\[ k_m = \Lambda \left(\zeta / 2\epsilon\right)^{1/2}, \quad \Sigma(k_m) = k_m^{2\zeta} e^{-\zeta} = k_m^{2\zeta} \mu^{\log \Lambda}. \]

The peak wavenumber thus scales as \( \epsilon^{-1/2} \), the same scaling as the dissipation scale.

The relative variance in that peak wavenumber scales as \( \epsilon^{-\zeta} \).

\( k_m \) largest wavenumber that must be included in a numerical calculation to capture the decay of variance correctly (fewer?).
Conclusions

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• Large-scale eigenmode dominates exponential phase, as for baker’s map. [Fereday et al., Wonhas and Vassilicos, PRE (2002)]
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- Large $K$? Periodic orbits?